TD n°10 (Martingales and Queues)

Exercice 1. *Aloha Stabilization*

Aloha is a communication protocol on a canal shared by several stations unaware of each other. Transmissions and retransmissions can only start at times of type $k\Delta$ with k integer and $\Delta > 0$ the width of a *slot*. When two stations try to transmit simultaneously messages, they interfere and none is actually transmitted. These *conflicts* are detected by stations. The protocol is the following :

- Fresh messages systematically try to pass right after their arrival.
- In case of conflict, each concerned station independently tries to retransmit its message at the next slot with probability $0 < \nu < 1$.

We denote by A_n the number of fresh messages arrived at the beginning of slot n and X_n the number of messages delayed at slot n. We assume that the r.v. A_n are i.i.d. and we set $a_i = \mathbb{P}(A_n = i)$, $\lambda = \mathbb{E}(A_n) = \sum_{i=0}^{\infty} i\, a_i$.

A - Aloha Instability :

1. Give the probability $b_i(k)$ that i stations try to retransmit if k stations are in conflict.

We will assume to simplify that the retransmission of a message depends only on itself and not on its station. This has the weird consequence that two messages from the same station can conflict. Under this assumption, $b_i(k)$ represents the probability that i messages are retransmitted if k ones are delayed.

2. Give the probability $p_{k,l}$ to pass from k to l delayed messages.

3. Show that this protocol is unstable (i.e., (X_n) is not positive recurrent).

4. What does it actually means for the protocol ?

B - Aloha Stabilization :

Instead of using a retransmission policy with ν fixed, we will try to reach stability using $\nu(k)$ depending on the number of delayed messages. We will show that the following condition implies stability.

$$\lambda < \lim\inf_{k\to+\infty} (b_1(k)a_0 + b_0(k)a_1)$$

It is equivalent to the existence of $\varepsilon > 0$ and a finite set $F \subset \mathbb{N}$ such that

$$\lambda < b_1(k)a_0 + b_0(k)a_1 - \varepsilon \quad \text{for all } k \notin F.$$

5. Under this assumption, prove the stability of the protocol.

6. Study the extrema of $g_k(\nu) = (1-\nu)^k a_1 + k\nu(1-\nu)^{k-1} a_0$.

7. Noticing that $\left(\frac{k-1}{k-a_1/a_0}\right)^{k-1} \xrightarrow{k\to\infty} \exp(\frac{a_1}{a_0} - 1)$, give a sufficient stability condition.

8. Explicit this condition when A_n follows a Poisson distribution.

9. What is the drawback of this policy ?

1 Exercise 2 : Garage

Exercice 2. *Garage (4 pts)*

A garage receives each day some cars to repair, V_n denotes the number of cars arriving during day n. Every day, one car among the cars arrived the days before is repaired and leaves the garage. From now on, we suppose that at the end of day 0, the garage is empty and that the sequence $(V_n)_{n\geqslant 1}$ is i.i.d. with $\mathbb{P}(V_1 = k) = p_k$, $k \in \mathbb{N}$.

1. Let us denote A_n the number of cars stored at the end of day n, show that $(A_n)_{n\in\mathbb{N}}$ is an homogeneous Markov chain and describe its transition graph or matrix.

2. Consider the particular case where $p_0 = p_1 = p_2 = 1/3$ and suppose that the garage has a maximum storage capacity M at night. What is the mean number of days for the garage to get overloaded, that is reaching strictly more than M cars stored at the end of the day?
Hint : you may use the first step analysis or a martingale of the form $X_n^2 + \alpha X_n + \beta n$.

3. Provide sufficient and necessary conditions about p_k, $k \in \mathbb{N}$ so that the chain $(A_n)_{n\in\mathbb{N}}$ is irreducible.

4. Assuming that the chain $(A_n)_{n\in\mathbb{N}}$ is irreducible, add some necessary and sufficient conditions about p_k, $k \in \mathbb{N}$ so that the chain is positive recurrent.

2 Exercise 3 : Discrete queues

Exercice 3. *Discrete queues (5 pts)*

In this exercise, time is discrete and we will study some small cases of discrete time queues combining discrete time poisson flows and markovian servers.

⊳ Our *markovian server* has an infinite buffer and a parameter q such that, at each time step, if the buffer is non empty or if it is empty and a packet arrives, then, with probability q and independently of everything else, the server instantaneously processes one packet which leaves the buffer.

⊳ A *discrete poisson* process is a traffic model with a parameter p such that, at each time step, one new packet arrives with probability p, independently of everything else.

⊳ A service policy is the way to choose the packet which is served first when several packets are waiting in the buffer. A *fixed priority server* assumes that each input flow has a different priority level (here a number) and that the server always chooses to process a packet with highest priority (here the lowest number).

1. Consider a fixed priority server with parameter 1 with two inputs flows. Flow 1 (resp. Flow 2) is a discrete Poisson process with parameter p_1 (resp. p_2) and priority 1 (resp. priority 2). Suppose that the buffer is empty at time 0 and consider X_n the number of packets from Flow 2 in the buffer at time n. Is it true that $(X_n)_{n\in\mathbb{N}}$ is a Markov chain? Explain briefly why and if it is markovian, describe the chain.

2. Consider a fixed priority server with parameter 1 with three inputs flows. For $1 \leqslant i \leqslant 3$, Flow i is a discrete poisson process with parameter p_i and priority i. Suppose that the buffer is empty at time 0 and consider X_n the number of packets from Flow 2 in the buffer at time n. Is it true that $(X_n)_{n\in\mathbb{N}}$ is a markov chain? Explain briefly why and if it is markovian, describe the chain.

3. Consider a fixed priority server with parameter 1 with five inputs flows. For $1 \leqslant i \leqslant 5$, Flow i is a discrete poisson process with parameter p_i and priority i. Suppose that the buffer is empty at time 0 and consider X_n the number of packets from Flow 3 in the buffer at time n. Is it true that $(X_n)_{n\in\mathbb{N}}$ is a markov chain? Explain briefly why and if it is markovian, describe the chain.

4. Consider a tandem system with two servers. Server 1 has parameter 0.5, Server 2 has parameter 0.4, their buffers are both empty at time 0. One discrete poisson flow of parameter 0.3 follows a path through Server 1 first and then Server 2. Study the dynamics of the system to answer the question : is it true that, from any state of the system, it will go back with probability 1 to the initial state where both buffers are empty?

5. Same question assuming that Server 2 has now parameter 0.2.