TD n° 1 - Some classical laws

Exercice 1. *Bernoulli and Binomial laws*

Let $p \in [0,1]$. A random variable X follows a *Bernoulli law* $\mathscr{B}(p)$ if X takes its values $\{0,1\}$ with $\mathbb{P}(X=1)=p$ and $\mathbb{P}(X=0)=1-p$.

1. By hand, compute the mean, the variance and the moment-generating series of X.

Let $p \in [0,1]$ and $n \in \mathbb{N}^*$. A random variable X follows a *binomial law* $\mathscr{B}(n,p)$ if X takes its values in $\{0,\dots,n\}$ and $\mathbb{P}(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$.

2. Give an example of a phenomenon following a binomial distribution $\mathscr{B}(n,p)$.

3. Compute the mean, the variance and the moment-generating series of X.

Exercice 2. *Poisson law*

Let $\lambda > 0$. A random variable X follows a *Poisson law* $\mathscr{P}(\lambda)$ if X takes its values in $\mathbb{N}$ and $\mathbb{P}(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$.

1. Give an example of a phenomenon following a Poisson distribution $\mathscr{P}(\lambda)$.

2. By hand, compute the mean, the variance and the moment-generating series of X.

3. Show that if X and Y are two independent random variables following Poisson laws of parameters λ and μ, then $X+Y$ follows the Poisson law of parameter $\lambda+\mu$.

The Poisson approximation principle states that the sum S_n of many independent Bernoulli random variables with small parameters almost follows a Poisson law of parameter $\mathbb{E}(S_n)$.

4. Illustrate this principle by proving that for $\lambda > 0$, any sequence of random variables $(X_n)_{n\in\mathbb{N}}$ respectively following a binomial law $\mathscr{B}(n,\lambda/n)$ converges in distribution (*Fr : convergence en loi*) to a Poisson law $\mathscr{P}(\lambda)$.

The *Poisson approximation principle* states that the sum S_n of many independent bernoulli random variables X_1,...,X_n with small parameters, tends to follow a Poisson law of parameter $\mathbb{E}(S_n)$.

5. 1000 competitors attend a fishing competition, each one having independently a probability 0,0001 to hook a fish and win a victory medal. The organizing commitee has bought 2 medals, is it enough to reward all the winners with probability at least 80 %?

Exercice 3. *Geometric law*

Let $p \in [0,1]$. A random variable X follows a *geometric law* $\mathscr{G}(p)$ if X takes its values in $\mathbb{N}^*$ and $\mathbb{P}(X=k)=(1-p)^{k-1}p$.

1. Give an example of a phenomenon following a geometric law $\mathscr{G}(p)$.

2. Compute the mean, the variance and the moment formal series for X.

3. Show that geometric laws are memoryless : for all $n,k \in \mathbb{N}$, $\mathbb{P}(X=k+n \mid X>n)=\mathbb{P}(X=k)$.

Exercice 4. *Exponential law*

Let $\lambda > 0$, a random variable X follows an *exponential law* $\mathrm{Exp}(\lambda)$ if X takes its values in $\mathbb{R}_+$ with density $f(x)=\lambda e^{-\lambda x}$.

1. Give an example of a phenomenon following an exponential law $\mathrm{Exp}(\lambda)$.

2. Compute the mean and the variance of X.

3. Show that exponential laws are memoryless : for all $a,b \in \mathbb{R}_+$, $\mathbb{P}(X \geq a+b \mid X \geq b)=\mathbb{P}(X \geq a)$.

4. Show the converse : any memoryless continuous law is exponential.

5. Let $X_1, \dots, X_n$ be independent exponential random variables of parameters $\lambda_1, \dots, \lambda_n$. Show that the random variable $\min(X_1, \dots, X_n)$ also follows an exponential law and find its parameter. Then, compute $\mathbb{P}(\min(X_1, \dots, X_n) = X_i)$ for $1 \leq i \leq n$. *Suggestion :* start with $n = 2$.

Exercice 5. *Normal law*

Let $m, \sigma^2 \in \mathbb{R}_+$, a random variable X follows a *normal law* $\mathcal{N}(m, \sigma^2)$ if X takes its values in $\mathbb{R}$ and has density $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$.

1. Give an example of a phenomenon following a normal law.

2. Compute the mean and variance of X.

3. Let X, Y be independent normal random variables of laws $\mathcal{N}(m_1, \sigma_1^2)$ and $\mathcal{N}(m_2, \sigma_2^2)$, show that $X + Y$ follows the law $\mathcal{N}(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

Exercice 6. *Will the sun rise tomorrow?*

A long time ago, a number p was chosen at random uniformly between 0 and 1, but this value was never revealed to mankind. Since this time, the sun rises every day with probability p (still unknown). What happened during the preceding days is independent of what happens today. You know that the sun has risen every day from the beginning, that is n times (and you know this number), what is the probability that it will rise tomorrow?