

Lecture 6 - Sensitivity analysis: Dual solutions and many other things...

Optimization and Approximation - ENS M1

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The basic idea of sensitivity analysis is very simple: the tableau gives you the optimal solution of the LP. But it actually gives MUCH MORE information on the structure of your solution ! The content of this lecture is devoted to give you some basic ideas of the information you can extract from a LP. More information can be derived. The interested reader is for referred to Chapter 3 of "Applied Mathematical Programming" for instance.

Let us first recall the Dovetail LP:

$$\max_{(x_1, x_2) \in \mathbb{R}^2} 3 \cdot x_1 + 2 \cdot x_2.$$

under the constraints

$$\begin{aligned} 3 \cdot x_1 + x_2 &\leq 18 \\ x_1 + x_2 &\leq 9 \\ x_1 &\leq 7 \\ x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

and the tableau of its optimal solution:

$$\begin{array}{cccc|cc|c} 0 & 0 & -\frac{1}{2} & -\frac{3}{2} & 0 & 0 & -\frac{45}{2} \\ \hline 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{9}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{3}{2} & 0 & 0 & \frac{9}{2} \\ 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & \frac{5}{2} \\ 0 & 0 & \frac{1}{2} & -\frac{3}{2} & 0 & 1 & \frac{3}{2} \end{array}$$

Optimal solution and value.

Let us first recall that you can read the optimal solution on the column at the right of the LP. For the initial variables, their values give you their value in the optimal solutions. For the slack / surplus variables, their value represent the distance between the optimal point and the hyperplane corresponding to the constraint. If the value equals 0, then the constraint is tight (and you can the optimal point is at the intersection of the tight constraints).

Shadow prices.

The *shadow price* associated with a particular constraint is the change in the optimal value per unit increase in the right-hand-side value for that constraint, all other problem data remaining unchanged.

Suppose that the wood capacity in the first constraint of our model

$$3x_1 + x_2 \leq 18$$

is increased from 18 to 19 cubic meters. We have an additional cubic meter of wood at no cost. So our new constraint in the standard LP becomes

$$3x_1 + x_2 + x_3 = 19 \Leftrightarrow 3x_1 + x_2 + (x_3 - 1) = 18$$

with a non negative x_3 . But we can also write it in the following way:

$$3x_1 + x_2 + x_3 = 18 \text{ and } x_3 \geq -1$$

And then, when we look at objective function, we realize that instead of staying at a position where $x_3 = 0$ we should push it to -1 (the coefficient is negative in the objective function, so decreasing x_3 without modifying x_4 improves the value of the solution). Note that since x_3 is a slack variable, it does not affect other constraints in the initial LP (x_3 only appears in that constraint).

What is the contribution to the optimal profit of this additional unit of capacity? If we set $x_3 = -1$ unit and keep x_4 to zero, we increase the value by $\frac{1}{2}$ unit.

So the shadow price of a constraint equals the opposite of its value in the objective vector. In particular, if a constraint is not tight, then its shadow price equals 0.

Right-hand ranges.

Is it possible to determine the coordinate of the new solution? And, second question, up to which point is it possible to know that this modification of the optimal solution is correct? And when thing changes, can we guess which variables are entering or leaving the basis? The answer to all these questions are positive.

Case 1: the slack / surplus variable is in the basis.

Let us illustrate this case on our example: the variable x_5 , which is the slack variable for the long matches boxes is in the basis. So if we replace the RHS = 7 at the beginning by the equality = $7 + \alpha$, then we can keep exactly the same set of constraints but write

$$x_1 + (x_5 - \alpha) = 7$$

rather than $x_1 + x_5 = 7 + \alpha$.

So by performing the same set of pivoting operations we finally obtain in our third row:

$$(x_5 - \alpha) + \text{Non Basic Term} = \frac{3}{2}$$

The variable x_5 remain positive (and we then have a feasible solution as long as $\alpha \in [-\frac{5}{2}, +\infty[$. In other words, if the RHS is between $\frac{9}{2}$ and $+\infty$, the basis is not modified. We can geometrically check that this is indeed correct since by pushing the equation to the right we do not add any constraint and if we push it to the left, we have to push (to the left) by its distance to the optimal point which is $\frac{3}{2}$.

Let us now try to determine which variable will enter and leave the basis. It is easy to determine which variable has to leave the basis. Indeed, if $\alpha < -\frac{3}{2}$ then the variable x_5 becomes negative. So x_5 has to leave the basis. Let us now explain how we can determine which variable is entering in the basis. We want to ensure that, after the pivot operation, all the coefficients of the objective function remain negative (to ensure that the solution is still optimal). So let us look at $z(t) = (\text{Obj Vector}) - t (\text{Row corresponding to } x_5)$, we have the following terms:

$$z(t) = \frac{1}{2}(-1 - t)x_3 + \frac{1}{2}(-3 + t)x_4 - tx_5 \tag{1}$$

And (i) one of these values must be equal to 0 (one variable must be in the basis) and (ii) all the other coefficients have to be non-positive (since otherwise the solution would not be optimal). And then in that case we have:

$$t = \frac{3}{2}$$

and the variable entering in the basis is x_4 .

Geometric interpretation. When the box constraint is pushed down, at some point, it should become tight. Then the variable is leaving the basis. If we continue to push, the machine constraint is not tight anymore, so the corresponding slack variable has to leave the basis.

In general, we can apply the same rule. If the slack/surplus variable x is in the basis. And we can then compute easily:

- The RHS interval for which the solution remains the same is an interval of the form $]-\infty, \alpha]$ or $[\alpha, +\infty[$,
- and we know that x is leaving the basis on α ,
- and we can determine which variable is entering in the basis on α .

Case 2: the slack / surplus variable is not in the basis.

Let us prove that we can make the same if the slack variable is not in the basis.

Let us illustrate this case on our example: the variable x_3 , which is the slack variable for the wood constraint which is not in the basis (in other words, the constraint is tight). As in the first case, the modification of the RHS by α can also be interpreted as ≥ 18 and the variable x_3 can be replaced by $x_3 - \alpha$. And it can be performed during all the algorithm. So in the final tableau we have the following tableau

$$\begin{aligned} x_1 + \frac{1}{2}(x_3 - \alpha) + \dots &= \frac{9}{2} \\ x_2 - \frac{1}{2}(x_3 - \alpha) + \dots &= \frac{9}{2} \\ x_5 - \frac{1}{2}(x_3 - \alpha) + \dots &= \frac{5}{2} \\ x_6 + \frac{1}{2}(x_3 - \alpha) + \dots &= \frac{3}{2} \end{aligned}$$

And, as in Case 1, the variable remains in the basis as long as the variables x_1, x_2, x_5, x_6 remain negative (indeed it is the best solution we can hope for since the non-basic terms remain equal to 0 and the value of the solution is a constant minus a positive function of the non-basic variables).

So x_3 remains out of the basis as long as:

$$\begin{aligned} -\frac{1}{2}\alpha &\leq \frac{9}{2} \\ \frac{1}{2}\alpha &\leq \frac{9}{2} \\ \frac{1}{2}\alpha &\leq \frac{5}{2} \\ -\frac{1}{2}\alpha &\leq \frac{3}{2} \end{aligned}$$

In other words, we have:

$$-3 \leq \alpha \leq 5.$$

And then the variables remains tight as long as the RHS remains in $[15, 23]$.

- The variable leaving the basis is the variable that becomes negative.
- The variable entering in the basis can be determine as in Case 1.

Pricing out new activities \Leftrightarrow Addition of variables

Is it the only information we can extract from the tableau? Not at all. Let us first show that, we can determine if a new activity can be interesting or not !

Assume that the "Dovetail" company has a new opportunity to produce toothpick. To produce 100,000 boxes of toothpicks, we need 1 cubic meters of wood and 100,000 small boxes. But the amount of time in the production machine (which can also produce toothpicks !) is twice the time necessary to produce short or long matches boxes. The profit for this new activity would be of 2,500\$ for 100,000 boxes of toothpicks. One can indeed add a new variable to the LP and solve it in order to determine if this activity is interesting or not. However, there is another option that permit via a simple calculation using the optimal tableau to determine if this activity is interesting or not.

In order to determine that, we have to consider the shadow prices of the constraints. Indeed, assume that we decide to produce toothpicks. How much money do we generate? The hypothesis tells us that we are generating 2,500\$ unit of profit. So we have to determine what is the "cost" of producing them. If we decide to produce one unit, we "lose one unit of wood and two units of production. We have seen just before that the opposite of the coefficients of the final objective function give the amount of money we gain / lose if the corresponding RHS is increased / decreased. So in that example, so it is interesting to produce toothpicks iff:

$$\frac{1}{2} \cdot 1 + \frac{3}{2} \cdot 2 \leq 2.5$$

So in that case, it is not interesting to start the new production ! And we can decide it via the computation of a single equation... We do not need to compute a new LP.

Solution of the dual.

The solution of the dual can be read on the final tableau: the optimal solution of the dual is $(\frac{1}{2}, \frac{3}{2}, 0, 0)$. Before explaining why, one can easily check that it satisfies all the constraints of the dual and that the value of that solution for the dual is $18 \cdot \frac{1}{2} + 9 \cdot \frac{3}{2} = \frac{45}{2}$. This solution is indeed optimal by the Strong Duality Theorem.

Generally speaking, in order to read the optimal solution of the dual (when the initial LP is in canonical form), we just have to read the coefficient in the objective function of the slack variable of the corresponding constraint to find its value in the optimal solution of the dual. This is not very surprising since we have seen that the shadow prices correspond to the benefit we get from an additional unit of resource... But then it is also the value we "price" one unit of resource, and then the minimum price at which we can accept to sell this resource.

And the price at which we accept to sell the resource exactly corresponds to the economic interpretation of the dual.

Dual simplex algorithm

Computing a solution of the dual then just consists in making operation on the rows rather than on the columns of the matrix. We know that opposite of the objective function row is a solution of the dual. So if at the beginning of Phase I, we do not have a solution of (P) but we have a solution of (D), in other words, we can find a solution for which all the "coefficients" are negative expressed in term of non-basic variables but some non-negativity constraints are violated, then we can little by little improve our solution by looking at some violated constraint (ie a negative coefficient in the last column) and modify the basis in order to get finally a solution where all the coefficients are positive. It consists in making moves in the dual up to obtain an optimal solution of it...