

Lecture 3 - Simplex algorithm

Optimization and Approximation - ENS M1

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1 Rewrite LPs: standard and canonical forms

Some conventions. Let us first give a few conventions concerning LP.

- If we consider a maximization problem and the function f is unbounded in F we say that the optimal value is $+\infty$. If F is empty we say that the optimal value is $-\infty$.
- If we consider a minimization problem and the function f is unbounded in F we say that the optimal value is $-\infty$. If F is empty we say that the optimal value is $+\infty$.

A Linear Program (or polytope) is in *canonical form* if its matrix constraint is of the form $Ax \leq b$, $x \geq 0$. It is usually one of the "simplest way" of understanding the shape of a polytope. Unfortunately, it is not the most convenient way of representing a LP from an algorithmic / programming point of view. Indeed, while inequalities are somehow hard to handle for a computer, equalities are much easier to catch. In what follow we define the standard form of a polytope and explain how we can obtain this LP from any LP.

1.1 Standard form of a linear program

A linear program can be a maximization or a minimization problem. The constraints can be represented using equalities or inequalities. The variables can be free or non negative. In this part we will show that all these formulations are equivalent since any formulation can be transformed into any other. We will introduce a *standard form* for a LP. This standard form is not necessarily the easier to read or understand but it is the one which is at the core of the simplex algorithm.

We first reorganize the inequalities with all the variables at the left-hand side (LHS) of the constraints and the constant terms at the right-hand side (RHS) of the constraints.

Inequality constraints. Let us first assume that we have the following constraints.

$$\sum_{j=1}^m x_{ij} \leq s_i \text{ for all } i \leq 2.$$

$$\sum_{i=1}^2 x_{ij} \geq d_i \text{ for all } j \leq m.$$

If we want to transform these inequalities into equalities, we create a new variable for each constraint. The variables y_i and z_j are defined as follows.

$$y_i = s_i - \sum_{j=1}^m x_{ij} \text{ for all } i \leq 2.$$

$$z_j = \sum_{i=1}^2 x_{ij} - d_i \text{ for all } j \leq m.$$

The variables y_i represent the difference between the available resources and the amount of resources used in practice. Such variables, coming from \leq inequalities, are called *slack variables* and represent the amount by which the right-hand side exceeds the left-hand side. On the other hand, the variables z_j represent the “excess” of production compared to the desired one. Such variables, which come from \geq inequalities, are called *surplus variables* and represent the amount by which the left-hand side exceed the right-hand side. Note that we construct these inequalities in such a way, if the constraints are satisfied then the slack and the surplus variables are non negative. Moreover, if the constraint is tight at some point x then the slack/surplus variable equals 0 at this point. Using slack and surplus variables, we can always transform inequality constraints into equality constraints in the following way:

$$y_i + \sum_{j=1}^m x_{ij} = s_i \text{ for all } i \leq 2.$$

$$\sum_{i=1}^2 x_{ij} - z_j = d_i \text{ for all } j \leq m.$$

where all the variables y_i and z_j are non-negative.

Free variables. Up to this point, we have only seen non-negative variables. In many cases, this constraint is satisfied. However it is not necessarily the case. A variable which is not necessarily non negative is called *free*. Assume that some LP has a free variable x . Then we can replace x by two variables x^+ and x^- and modify the LP in the following way:

- Each occurrence of x is replaced by $x^+ - x^-$.
- We add two non-negativity constraints $x^+, x^- \geq 0$.

Positivity of the RHS. If a constraint has a negative RHS after these transformations, we multiply both side of the equality by -1 in order to obtain a non-negative RHS.

Example of Dovetail. Let us transform our favorite example of “Dovetail” into the standard form. Recall that the LP is:

$$\max_{(x_1, x_2) \in \mathbb{R}^2} 3 \cdot x_1 + 2 \cdot x_2.$$

under the constraints

$$\begin{aligned} 3 \cdot x_1 + x_2 &\leq 18 \\ x_1 + x_2 &\leq 9 \\ x_1 &\leq 7 \\ x_2 &\leq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

There are no free variables. So we just have to add slack variables. The resulting constraints are the following:

$$\begin{array}{rcccccc} 3 \cdot x_1 + x_2 & + x_3 & & & & = 18 \\ x_1 + x_2 & & + x_4 & & & = 9 \\ x_1 & & & + x_5 & & = 7 \\ x_2 & & & & + x_6 & = 6 \end{array} \quad x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

Or in matrix notation we have:

$$\begin{pmatrix} 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \\ 7 \\ 6 \end{pmatrix}$$

The feasible region of this linear program seems more complicated than the feasible region of the original problem, but this is not true: the feasible region has the same “shape”. We have not created new solutions in the sense that the “projection” of a solution of the new LP provides a solution of the original LP.

1.2 Basic and nonbasic variables

Remark 1: Basic Feasible Solutions and canonical form. Assume that we have a polyhedron that is given in its canonical form (i.e. only with \leq constrains and non negative variables). Assume that the polyhedron has m constraints (except the non negativity variables) and n variables. Then a basic feasible solution is a point that is tight for at least n constraints. Note that some of them may be the non-negativity constraints

So when we consider the standard form the corresponding LP, at least n variables of the corresponding solution are equal to 0. Indeed:

- If a constraint is tight, then its slack variable is equal to 0.
- If an initial variable equals 0, then it is indeed equal to 0...

Exercise 1. Prove that a solution is degenerate if and only if more than m variables are equal to 0 in the corresponding solution in the LP in standard form.

Basic variables In the previous paragraph, we have seen that in a LP in standard form, it suffices to choose m variables that might not be equal to zero and that the other can be put to zero. So in what follows, we will have a set of m basic variables and n non-basic ones. The goal would consist in finding the basic feasible solution corresponding to this set of basic variables and express the objective vector in terms of the non-basic ones.

Before defining formally all these notions, let us illustrate them on the “Dovetail” example.

Illustration on our example. We have a linear system with 6 variables and only 4 equations. Each vertex of the polyhedra corresponds in the initial LP at the intersection of two “equality” constraint (since the space of variable has dimension two). So, in other words, it corresponds to set two variables to zero and then try to find (if it exists) the point of intersection of the four remaining constraints (of the new LP). Note that since we consider affine spaces, the intersection of these four affine spaces can be empty (for instance if two remaining constraints are “parallel”).

Let us choose a set of variables that are to be set to zero, say $NI = \{3, 4\}$. The variable x_i is put to zero if $i \in NI$. Let $BI = \{1, 2, 3, 4, 5, 6\} \setminus NI$. The sets NI and BI partition the set of variables. The set NI is said to be the set of nonbasic variables and the set BI is called the set of basic variables. If we put the nonbasic variables on the right hand side, we can rewrite the constraints of the model as:

$$\begin{array}{rcll} 3 \cdot x_1 + x_2 & = & 18 & -x_3 \\ x_1 + x_2 & = & 9 & -x_4 \\ x_1 & + & x_5 & = 7 \\ x_2 & + & x_6 & = 6 \end{array}$$

Or in matrix notation:

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \\ 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$$

Let us denote by B the 4×4 matrix of the left hand side (B stands for basic), x_B the vector of the left hand side. We denote by b the vector of constraints, by N the non-basic matrix (where N stands for non-basic) and by x_N the vector of the right hand side.

Since B is invertible (i.e. B^{-1} exists, i.e. $Bx_B = 0 \Rightarrow x_B = 0$), we can invert this matrix and determine x_B in function of x_N . More formally we have:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 18 - x_3 \\ 9 - x_4 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{pmatrix} \begin{pmatrix} 18 - x_3 \\ 9 - x_4 \\ 7 \\ 6 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 9 - x_3 + x_4 \\ 9 + x_3 - 3x_4 \\ 5 + x_3 - x_4 \\ 3 - x_3 + 3x_4 \end{pmatrix}$$

This means that the basic variables can be expressed in terms of the nonbasic variables. So, once we have fixed the values of the nonbasic variables, we know the values of the basic variables. In particular, if we set our nonbasic variables to zero, i.e. $x_3 = x_4 = 0$, then we find $x_1 = x_2 = \frac{9}{2}$ (as we expect! why?) and $x_5 = \frac{5}{2}$ and $x_6 = \frac{3}{2}$. Note that it means that when the constraints 1 and 2 are tight (since the difference between the LHS and the RHS equal zero). The difference between the LHS and the RHS in constraint 3 is $\frac{5}{2}$ and the difference in constraint 4 is $\frac{3}{2}$. In particular these two last constraints are non-binding. On the opposite, the constraints 1 and 2 are binding (since x_3 and x_4 are equal to zero).

Note that we can also (and it is a second huge advantage of the simplex algorithm) state our objective function in terms of the nonbasic variables:

$$3x_1 + 2x_2 = \frac{3}{2} \cdot (9 - x_3 + x_4) + \frac{2}{2}(9 + x_3 - 3x_4) = \frac{45}{2} - \frac{1}{2}x_3 - \frac{3}{2}x_4$$

Recall that we are trying to maximize this function. This expressions states that if we are “currently” at the vertex $(\frac{9}{2}, \frac{9}{2})$, and we consider increasing x_3 , then for every unit by which we increase x_3 , the objective value will decrease by $\frac{1}{2}$ units. Likewise, for every unit by which we increase x_4 , the objective value decreases by $\frac{3}{2}$ units. So, since $x_3, x_4 \geq 0$, this expression states that we are currently in a local maximizer of our objective function.

Since we have seen that a local maximizer of a LP also is a global optimizer, the point $x_1 = x_2 = \frac{9}{2}$ provides a global optimizer. And then, if we choose $x_3 = x_4 = 0$, our “current” vertex is optimal.

Note that the coefficients of x_3 and x_4 in the right hand side are the shadow prices of the constraints 1 and 2 respectively. Let us informally explain why: consider the constraint 2. Assume that we increase the productivity of the machine by 1 unit. It is equivalent to say that the constraint on x_4 is $x_4 \geq -1$ unit instead of $x_4 \geq 0$. And then our expression ensures that the objective function increases by $\frac{3}{2}$ units, which is precisely what we obtained.

2 Improvement of solutions: Pivoting

In the previous lecture, we have seen that when we choose the “best” basis, then we obtain the optimal solution. Let us now explain how we can improve the solution when we do not choose the best basis at the beginning.

In the “Dovetail” example, a natural basis is the basis $BI = \{3, 4, 5, 6\}$. Indeed the matrix B which is the matrix $(A \quad I_m)$ restricted to the last 4 columns is the matrix I_4 . So in particular, the matrix is invertible. Moreover since $B = B^{-1} = I_4$ and since $b \geq 0$, we have $B^{-1}b \geq 0$. So BI is a basic feasible solution. And we have $NI = \{1, 2\}$. In particular we have:

$$x_B = b - Nx_N$$

In the following we express the basic variables in terms of the nonbasic variables. So we have:

$$\begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 - 3x_1 - x_2 \\ 9 - x_1 - x_2 \\ 7 - x_1 \\ 6 - x_2 \end{pmatrix}$$

Because of this set of equalities, we can easily express the objective function in terms of the nonbasic variables (which is already the case): $\max 3x_1 + 2x_2$.

So it seems profitable to increase x_1 or x_2 in order to obtain a better solution (note that the current solution puts both x_1 and x_2 to zero, so the current solution equals zero which is obviously not optimal).

Let us try to increase x_1 . How far can we increase x_1 ? Clearly if x_1 increases too much, we leave the feasible region. In order to determine the “limits” of the growth of x_1 , let us look at the constraints and determine up to which point x_1 can be increased without violating the non-negativity constraints. We have:

$$\begin{aligned} x_3 = 18 - 3x_1 - x_2 \geq 0 & \Rightarrow x_1 \leq 6 \\ x_4 = 9 - x_1 - x_2 \geq 0 & \Rightarrow x_1 \leq 9 \\ x_5 = 7 - x_1 \geq 0 & \Rightarrow x_1 \leq 7 \\ x_6 = 6 - x_2 \geq 0 & \text{has no constraint on } x_1. \end{aligned}$$

So x_1 can be increased up to $\min(9, 6, 7) = 6$ without violating any constraint. This test is usually called the *ratio test*. When x_1 is increased up to 6, x_3 decreases to 0. Differently, when $x_1 = 6$, two variables are equal to 0: x_3 and x_2 . So we have a new basic solution: the variable x_1 is added in *BI* (and deleted from *NI*) and x_3 is added in *NI* (and deleted from *BI*). This operation is called a *pivoting* or a *pivot operation*. The modification of the basis modify the equation system:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \\ 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 18 - x_2 - x_3 \\ 9 - x_2 \\ 7 \\ 6 - x_2 \end{pmatrix}$$

At this point we should compute the invert of the current matrix B in order to determine x_1, x_4, x_5, x_6 in terms of the non-basic variables.

$$\begin{aligned} x_1 &= 6 - \frac{1}{3}x_2 - \frac{1}{3}x_3 \\ x_4 &= 3 - \frac{2}{3}x_2 + \frac{1}{3}x_3 \\ x_5 &= 1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \\ x_6 &= 6 - x_2 \end{aligned}$$

And the objective function can be written in terms of the nonbasic variables as: $\max 3(6 - \frac{1}{3}x_2 - \frac{1}{3}x_3) + 2x_2 = 6 - x_2 - x_3 + 2x_2 = 18 - x_3 + x_2$.

So the “current basic solution” gives a solution of 18 (which is not optimal since we know that the optimal value is 22.5). But we can see that if we make a pivoting operation on x_2 then we will increase the value of the solution. Indeed, increasing x_3 can only decrease the value of the solution but increasing x_2 can only increase this value.

So as in the previous case, we want to determine up to which point x_2 can be increased.

$$\begin{aligned} x_1 = 6 - \frac{1}{3}x_2 - \frac{1}{3}x_3 \geq 0 & \Rightarrow x_2 \leq 6 \cdot 3 = 18 \\ x_4 = 3 - \frac{2}{3}x_2 + \frac{1}{3}x_3 \geq 0 & \Rightarrow x_2 \leq 3 \cdot \frac{3}{2} = \frac{9}{2} \\ x_5 = 1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 \geq 0 & \Rightarrow \text{no restriction on } x_2 \\ x_6 = 6 - x_2 \geq 0 & \Rightarrow x_2 \leq 6 \end{aligned}$$

So x_2 can increase up to $\frac{9}{2}$ and this is tight for the constraint 2. So the variable x_4 and x_2 can be exchanged in the basis. More precisely we can add x_2 in *BI* and we can add x_4 in *NI*. So at this point we have

$$BI = \{1, 2, 5, 6\} \quad \text{and} \quad NI = \{3, 4\}$$

We precisely considered that case at Lecture 1 and then a sequence of pivoting rules lead to the optimal solution (do it).

3 Simplex tableau

We have developed a general method (pivoting) which ensures that we can improve our solution at any step. Let us now explain briefly how we can represent the input of a LP into a computers. We will represent LPs with *Simplex Tableaux*.

Assume that we have a basis BI and let NI be the decision variables which are not in BI . Our simplex tableau will contain $m + 1$ rows where m is the number of constraints and the additional row will represent the objective function. When we represent we will put the objective function on the first row and separate this row from the others with a horizontal line. The columns of the simplex tableau correspond to the decision variables. There is an additional row which represents the vector b . We assume that we have a natural order on the constraints and on the decision variables: the i -th row corresponds to the i -th constraints and the i -th column corresponds to the i -th variable.

The coefficient $a_{i,j}$ of the array is the coefficient of x_j in the i -th constraint. The j -th coefficient a_j of the row of the objective function is the coefficient of x_j in the objective function and has to be equal to zero if x_j is a basic variable (we have seen that it is possible to do that). Moreover, if x_j is in the basis then the column of j must contain precisely one nonnegative entry which has to be equal to one. In other words, the submatrix restricted to the columns of the basic variables has to be "isomorphic" to the identity matrix. In the last column of the row of the objective function, we put the current value of the objective function.

Let us illustrate the example of the previous section.

$$\max 3x_1 + 2x_2$$

under

$$\begin{pmatrix} 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 9 \\ 7 \\ 6 \end{pmatrix}$$

At the beginning we have chosen $BI = \{3, 4, 5, 6\}$. Note that the matrix restricted to the variables of the basis is already isomorphic to the identity matrix. Moreover the objective function is written in terms of the non-basis variables. So the matrix and the objective function already have the good shape. And the Simplex tableau can be written in the following way:

$$\begin{array}{cccccc|c} 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 3 & 1 & 1 & 0 & 0 & 0 & 18 \\ 1 & 1 & 0 & 1 & 0 & 0 & 9 \\ 1 & 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & 1 & 6 \end{array}$$

In order to improve the solution, we can add x_1 or x_2 in the basis. Note that the ratio test for x_1 can be done on the Simplex tableau by just dividing for each constraint the value of the last column by the coefficient of x_1 in the row and keeping the smallest value (for instance for x_1 and for the first constraints, we can increase x_1 up to $18/3 = 6$).

Let us illustrate on this simplex tableau the modification of the basis into $BI = \{x_1, x_4, x_5, x_6\}$. As before, we proceed by making a *Jordan Gauss elimination* (we subtract one third of the second rows to the third and fourth columns). It gives:

$$\begin{array}{cccccc|c} 3 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 6 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 & 0 & 3 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 6 \end{array}$$

We do not have yet a simplex tableau since the row of the objective function has not been modified: we have to express x_1 in function of the non basic variables. And so we have:

$$\begin{array}{cccccc|c} 0 & 1 & -1 & 0 & 0 & 0 & -18 \\ \hline 1 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 6 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 1 & 0 & 0 & 3 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 6 \end{array}$$

Simplex tableaux give a nice way to represent in a compact way LP. In the following, we will essentially use this notation to represent linear programs.

4 Termination criterion

Assume that the Linear Programming is written in the following way:

$$\begin{aligned} & \max c_N^T x_N + S \\ (I \quad A) \begin{pmatrix} x_B \\ x_N \end{pmatrix} &= b \\ x &\geq 0 \end{aligned}$$

where $b \geq 0$, x_B is the vector of a basic feasible solution. So in particular, when $x_N = 0$, the we can obtain a feasible solution of value S . We can stop when:

1. If we have $c_N \leq 0$ and then the optimal value equals S .
2. If there exists a coefficient c_i of c_N such that c_i is positive and every coefficient of the column of x_i in $(I \quad A)$ are non-positive. And then the solution is infinite

In what follows, we prove that the claim above is true.

4.1 Point 1: finite optimal value

Let $x_B = b$ and let $x_N = 0$. We have

$$(I \quad A) \begin{pmatrix} x_B \\ x_N \end{pmatrix} = b.$$

Moreover the value of this solution is S . Now assume that there exists a better solution $y = \begin{pmatrix} y_B \\ y_N \end{pmatrix}$. Since $y_N \geq 0$ and since $c \leq 0$, we have $c_N^T y_N \leq S$, contradicting the fact that it is a better solution.

4.2 Point 2: infinite optimal value

Assume now that there exists a coefficient of c_N which is positive, say it corresponds to the variable x_i . Assume moreover that any coefficient of the column of A corresponding to x_i , there are only non-positive values. Note that the form of the matrix ensures that there exists a variable x_j such that x_j appears in only one row, which is the row j . Consider the vector x_N^M such that for every $j \in NI \setminus i$, $x_j = 0$ and such that $x_i = M$. Now consider the vector x_B such that for every $j \in BI$ we have $x_j = b_j - a_{j,i}M$ (where $a_{j,i}$ is the coefficient of the j -th row in the column corresponding to i). Note that the vector x_B is nonnegative since b is non negative, M is non negative and $a_{i,j}$ is non-positive. So the vector $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$ satisfies the constraint.

The objective function strictly increases (linearly) when M increases. So if M is arbitrarily large, the objective function grows arbitrarily. And then the objective function is unbounded.

Let us see on some example what really happens in this case. Consider the following LP:

$$\max(2x_1 + x_2)$$

Subject to:

$$\begin{aligned} x_1 - x_2 &\leq 10 \\ 2x_1 - x_2 &\leq 40 \\ x_1, x_2 &\geq 0. \end{aligned}$$

The addition of slack variables permits to write the LP in standard form. The two slack variables are called x_3 and x_4 and the corresponding tableau is

$$\begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 10 \\ 2 & -1 & 0 & 1 & 40 \end{array}$$

Let us first put x_1 into the basis. One can note that x_3 leaves the basis (since $x_1 \leq 10$ is given by the first constraint). We obtain the tableau:

$$\begin{array}{cccc|c} 0 & 3 & -2 & 0 & -20 \\ 1 & -1 & 1 & 0 & 10 \\ 0 & 1 & -2 & 1 & 20 \end{array}$$

Since x_2 has a positive coefficient in the objective function, we have not reached the optimal value. This pivoting operation ensures that x_2 enters in the basis and x_4 leaves the basis.

$$\begin{array}{cccc|c} 0 & 0 & 4 & -3 & -80 \\ 1 & 0 & -1 & 1 & 30 \\ 0 & 1 & -2 & 1 & 20 \end{array}$$

Again the tableau is not optimal. Indeed the coefficient of x_3 is positive. So we want to apply a pivot operation on x_3 (the coefficient of x_4 is negative so it is not possible to pivot on x_4). But the problem is the following: the constraints do not give any constraint on x_3 . Indeed:

$$-x_3 \leq 30 \Rightarrow x_3 \geq -30$$

$$-2x_3 \leq 20 \Rightarrow x_3 \geq -40$$

So it means that, if x_3 enters in the basis, then we can arbitrarily increase x_3 without violating any constraint. Since the coefficient in front of x_3 is positive, it means that we can arbitrarily increase the value of the objective function without violating constraints. Thus the optimal value is unbounded!

More formally, what we have is that for any nonnegative δ , the point $(30 + \delta, 20 + 2\delta, \delta, 0)$ is feasible. Since the objective function at this point has value $80 + 4\delta$, we see that the problem is unbounded.

Clearly, unboundedness of a problem can occur only when the feasible region is unbounded, which, unfortunately, is something we cannot tell in advance of the solution attempt. In the above example, we detected unboundedness when we encountered a pivot column that does not contain any positive entry. More generally, we can in fact declare a problem as unbounded if any (nonbasic) column, not necessarily associated with the entering variable, is identified to have the above-stated property at the end of an iteration. Referring back to the initial tableau, we see that, indeed, the x_2 -column had this property. Therefore, we could have concluded that the problem is unbounded at the outset.

Note finally that if the function is a minimization function, then the criterion are reversed.

5 Simplex algorithm

A scheme of algorithm. We choose arbitrarily a set of nonbasic decision variables, and called the other variables basic. Then, we rewrote the system of constraints by putting the nonbasic variables in the right hand side. More formally, in order to construct the new system of constraints we have just done the following:

- We keep the columns corresponding to the basic variables on the left hand side.
- We move the columns corresponding to the nonbasic variables on the right hand side.

Then we have inverted the matrix to write the basic variables in function of the nonbasic variables. For doing that we have to be sure that the set of basic variables induces an invertible matrix. So what was really important in our choice of basic and nonbasic variables was not the number of nonbasic variables, but the number of basic variables. We choose as many basic variables as there are constraints: in this case we have a chance that B is invertible. (Recall that if a matrix is not squared, then it is not invertible).

After this transformation, we do the following:

- We check that all the variables are non-negative (hopefully it is the case there).
- We express the objective function in terms of the nonbasic variables.

If *all* the elements of the objective vector are negative, then we have a local maximizer, which then is a global maximizer. Indeed, it means that if we modify locally a bit our nonbasic variables (which are minimized in the current point) then we can only increase the non-basic variables and then decrease the value of the solution. On the opposite, if one element of the objective vector is positive, then it means that we have a “direction to follow”. Indeed, if we increase the value of the nonbasic variable, then we can increase the value of the solution. So we will try to put this variable in the set of basic variable and put another variable in the set of nonbasic variables. This operation is called a *pivot*.

Thus we can repeat this produce by modifying the set of nonbasic variables until we find an optimal solution. However we need an initial feasible solution in order to start this algorithm.

A few advantages of the Simplex Algorithm. The simplex algorithm will take as data a LP in standard form (called canonical form in Bradley, Hax, Magnanti) and solve it. As we already mentioned, the Simplex Algorithm will consist in looking for a local maximizer which is also a global maximizer. The simplex algorithm has many advantages:

- It is *robust* since it can solve any linear program (unbounded linear programs or linear programs with empty feasible regions). Moreover it can detect redundant constraints.
- It is *self-initiating* since it uses itself to generate an initial feasible solution or show that the problem does not have any solution.
- It provides *more information than the optimal solution*. It indicates how the optimal solution is modified in function of the data (sensitivity analysis).

Before introducing the simplex algorithm, we need the following notion of standard form.