TD1 - Modelization and Linear Programming

Indication of hardness: from (*) to (****).

1 *Modelization*

Exercise 1 - Lumberjack (*)
A lumberjack has 100 hectares of hardwoods. Cutting an hectare of wood and letting the zone regenerate naturally cost $10k\$$ per hectare and the benefit is $50k\$$. Alternatively, cutting an hectare of wood and re-seeding with pine-woods costs $50k\$$ per hectare, and the benefit is $120k\$$. We assume that the lumberjack has only $4000k\$$ funds in the beginning of the operation.
(a) Formulate the problem as a LP.
(b) Represent the LP geometrically.
(c) Find the best strategy to adopt and the benefit associated.

Exercise 2 - (Student) diet problem (*)
The diet problem consists in finding, at minimum cost, what can be eaten in order to satisfy a set of daily nutritional requirement. The quantity of kilo-calories required each day is between 2500 and 3500, and the quantity of fat is between $80g$ and $120g$.

Help the (moneyless) following student to optimize his next shopping at the supermarket. His goal is indeed to spend less money to buy more beer this week-end.

	Calories	Fat
Instant Noodles	460	21
Pizzas	700	20
White rice	400	0

A portion of instant noodles cost 0.5 euros, pizzas 3.5 euros and rice 0.5 euros.
(a) Formulate the problem as a Linear Program.
(b) Solve it geometrically. Is it cheaper for the student to go to the "*Restaurant Universitaire*" ?

Exercise 3 - Bank allocation (*)
A bank gets 1000000 euros to give via loans to its clients. It can loan money for (i) consumer credit, (ii) car credit and (iii) mortgages. Interests 6% for (i), 4% for (ii) and 2% for (iii). Moreover, the bank has (officially) to respect some rules:

- Consumer credit cannot represent more than 30%.
- Mortgages must represent at least 40%.
- The average rate cannot exceed 3.2%.

Formulate the problem as a Linear Program and solve it.

Exercise 4 - School (*)
A school director wants to change all desks in all classrooms. He managed to obtain tabletops at a good price and now has to buy metal rods to build the legs. The company offers him cheap rods of 2.10 meters. Depending on the age of the kids (and especially on their height!), the height of the desks varies. The director wants to build 60 small desks (height: 50cm), 80 medium desks (height: 80cm) and 65 high desks (height: 1.10m). Hence, to build a small desk he has to cut 4 rods of 50cm. The director wonders how to cut the rods of 2.10m to obtain the required number of legs minimizing the scrap metal. Translate this problem into a linear problem.

Exercise 5 - Custom molder (*) (from *Applied Math. Prog.* (Bradley, Hax, Magnanti))

Suppose that a custom molder has one injection-molding machine and two different dies to fit the machine. Due to differences in number of cavities and cycle times, with the first die he can produce 100 cases of six-ounce juice glasses in six hours, while with the second die he can produce 100 cases of ten-ounce fancy cocktail glasses in five hours. He prefers to operate only on a schedule of 60 hours of production per week. He stores the week's production in his own stockroom where he has an effective capacity of 15, 000 cubic feet. A case of six- ounce juice glasses requires 10 cubic feet of storage space, while a case of ten-ounce cocktail glasses requires 20 cubic feet due to special packaging. The contribution of the six-ounce juice glasses is $ 5.00 per case; however, the only customer available will not accept more than 800 cases per week. The contribution of the ten-ounce cocktail glasses is $ 4.50 per case and there is no limit on the amount that can be sold. How many cases of each type of glass should be produced each week in order to maximize the total contribution?

2 Discrete Mathematics and LP

Exercise 6 - Independent Set Problem (*)
Given a graph $G = (V, E)$, an *independent set* is a set of vertices pairwise not incident. Formulate the Maximum Independent Set problem as an ILP.

Exercise 7 - SAT (*)
Represent the SAT problem as a ILP problem. What can you deduce on the complexity status of ILP problems?

Exercise 8 - Dominating Set Problem (*)
Given a graph $G = (V, E)$, a *dominating set* is a set of vertices X such that any vertex of the graph is either in X or incident to a vertex of X.
Formulate the minimum Dominating Set problem as an ILP

Exercise 9 - N-queens problem (**)
Suppose that you are given an $N \times N$-chessboard.

(a) Formulate the problem of placing N queens on the board such that no two queens share any row, column or diagonal as an Integer Linear Programming problem.

(b) Code it to find a solution (if it exists) on an 8×8 chessboard.

Exercise 10 - TSP (****) The TSP problem is the following problem. We are given a complete graph G where edges are weighted. The goal consists in finding a cycle passing exactly once through all the vertices of the graph such that the sum of the weight is minimized.

Formulate this problem as an ILP where there is one variable per edge.

Hint: The ILP might have an exponential number of constraints.

3 Approximation

Exercise 11. Knapsack [Midterm 2018] (**)
A thief, who has a backpack with a capacity of 60 liters, is confronted with the problem of choosing the objects to steal among seven possibilities. Respective volumes and sell prices are given in the following table:

	Object 1	Object 2	Object 3	Object 4	Object 5	Object 6
Volume	20	16	7	10	30	12
Price	25	18	10	12	45	14

(a) Formulate this problem as an ILP.

(b) Assume now that there are m items. Is the optimal value of the fractional relaxation always equal to the optimal value of the ILP? Proof or counterexample.

(c) Assume that the volume of every item is smaller than the capacity of the backpack. Propose a 2-approximation algorithm.

Exercise 12 - Hitting Set (**)
A *k-hypergraph* is a pair (V, E) where V is a set of vertices and E is a set of (non ordered) subsets of vertices of size k. Each of these subsets of size k is a hyperedge. A *hitting set* is a set of vertices intersecting every hyperedge. The hitting set problem consists in finding the smallest possible Hitting Set.

(a) What is the Hitting Set problem for 2-hypergraphs?

(b) Formulate the Hitting Set problem as an ILP.

(c) Give a k-approximation algorithm for the Hitting Set problem on k-hypergraphs.

Exercise 13 - Grundy coloring and approximation (****)
Let $G = (V, E)$ be a graph and $v_1, \ldots, v_n$ be an order σ on the vertices. The *grundy coloring* consists in coloring the vertices in the order of σ in such a way v_i is colored with the smallest possible color (ie the smallest color that does not appear in $N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$. The number of colors used to color G with σ is denoted by $\mathcal{G}(G, \sigma)$. The grundy value of G is $\max_{\sigma \in \pm} \mathcal{G}(G, \sigma)$.

(a) Show that there exists σ such that $\mathcal{G}(G, \sigma)$ is the chromatic number of G.

(b) Show that there exists trees of arbitrarily large grundy value

Hint for (b): Prove it by induction.

4 The world is linear...

Exercise 14 - Transformation into LP (**)
Suppose that (P) is a program with variables x_1, x_2 with linear constraint and the following objective function. Explain how to modify (P) to get a linear program. If needed, you can add variables and constraints.

(a) $z = \min(\max(x_1, x_2))$.

(b) $z = \min |x_1|$.

(c) $z = \min(|x_1| + |x_2|)$.

(d) $z = \min(f(x_1))$ where $f(x_1) = x_1$ on $[0, 10]$, $f(x_1) = 0$ if $x_1 \leq 0$ and equals 10 otherwise.

Exercise 15 - Simple LP (*)
Find a LP of the shape $\max c^t x$ under $Ax \leq b$ as simple as possible such that:

(a) It does not have any solution $Ax = b$.

(b) The optimal value is infinite.

(c) The optimal solution is not unique.

Exercise 16 - LP and ILP (***)
Find a feasible LP where the constraint matrix is a $0 - 1$-matrix, the constraint and objectives vectors are $0 - 1$-vectors and where the gap between the optimal integral value and optimal real value is arbitrarily large.

Hint: There are many options. You can for instance consider the general hitting set problem.

5 The world is not linear...

Exercise 17 - Cylinders (***)
Find a cylinder with a given surface A which has the largest volume V.

(a) Express this problem as an optimization problem. Is it a linear programming problem?

(b) Give the optimal solution.