

# Linear Algebra

[KOMS119602] - 2022/2023

## 5.2 - Determinants of Matrices

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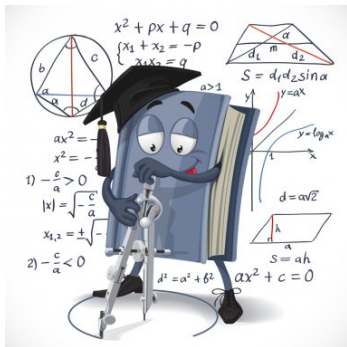
Week 7-11 February 2022

# Learning objectives

After this lecture, you should be able to:

1. implement the properties of determinant in problem solving;
2. compute determinant of matrix using cofactor expansion;
3. solve a system of linear equations using the Cramer's rule;
4. explain the procedure of computing the determinant of a diagonal block matrix.

*Good math skills are developed by doing lots of problems.*



# Part 5: Properties of determinant

# Determinant of the matrix transpose

## Theorem

$$\det(A^T) = \det(A)$$

**Q:** Can you explain why? Check it for the  $2 \times 2$  matrix and the  $3 \times 3$  matrix.

## Implication:

*Any theorem about the determinant of a matrix  $A$  that concerns the rows of  $A$  will have an analogous theorem concerning the columns of  $A$ .*

# Basic properties of determinant

Theorem (Determinants related to the *shape* of the matrix)

Let  $A$  be a square matrix.

1. If  $A$  has a row (column) of zeros, then  $|A| = 0$ .
2. If  $A$  has two identical rows (columns), then  $|A| = 0$ .
3. If  $A$  is triangular (i.e.,  $A$  has zeros above or below the diagonal), then  $|A| = \text{product of the diagonal elements}$ :

$$|A| = \prod_{i=1}^n a_{ii}$$

Particularly, for the identity matrix  $I$ , we have  $|I| = 1$ .

**Q:** Give an argument explaining why those properties hold!

# Elementary operations and determinant

## Theorem (Determinants & Elementary Row Operations)

*Suppose  $B$  is obtained from  $A$  by an elementary row (column) operation.*

- 1. If two rows (columns) of  $A$  were interchanged, then  $|B| = -|A|$ .*
- 2. If a row (column) of  $A$  were multiplied by a scalar  $k$ , then  $|B| = k|A|$ .*
- 3. If a multiple of a row (column) of  $A$  were added to another row (column) of  $A$ , then  $|B| = |A|$ .*

**Q:** Give an argument explaining why those properties hold!

# Determinant of matrix product

## Theorem (Determinant of matrix product)

*Given two square matrices  $A$  and  $B$  of the same order. Then:*

$$\det(AB) = \det(A) \cdot \det(B)$$

**Proof:** [https:](https://proofwiki.org/wiki/Determinant_of_Matrix_Product)

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An **elementary matrix**  $E_n$  is a matrix which differs from the identity matrix  $I_n$  by one single elementary row operation.

## Corollary

If  $E$  is an elementary matrix of size  $n$ , and  $A$  is an  $n \times n$  square matrix. Then  $|EA| = |E||A|$ .

# Exercise

Given:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$$

1. Compute  $AB$
2. Compute  $\det(A)$ ,  $\det(B)$ , and  $\det(AB)$ .
3. Is it true that  $\det(A) \cdot \det(B) = \det(AB)$  ?

## Exercise

Given:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$$

1. Compute  $AB$

$$AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

2. Compute  $\det(A)$ ,  $\det(B)$ , and  $\det(AB)$ .

$$\det(A) = 1, \quad \det(B) = -23, \quad \det(AB) = -23$$

3. Is it true that  $\det(A) \cdot \det(B) = \det(AB)$  ?

# Part 6: Computing determinant by cofactor expansion (*an algorithmic approach*)

## Minors and cofactors

Let  $A = [a_{ij}]$  be an  $n$ -square matrix.

Let  $M_{ij}$  be the  $(n - 1)$ -square matrix obtained from  $A$  by deleting the  $i$ -th row and the  $j$ -th column of  $A$ .

The **minor of the element  $a_{ij}$  of  $A$**  is defined as:

$$\text{minor}(A) = \det(M_{ij})$$

The **cofactor of  $a_{ij}$**  is defined as the **signed minor** of  $a_{ij}$ , and denoted by:

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

The pattern of the sign minor of elements in  $A$  can be written as:

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

## Example: *minors and cofactors*

Given matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the minor and the cofactor of the element  $a_{32}$ !

## Example: *minors and cofactors*

Given matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the minor and the cofactor of the element  $a_{32}$ !

### **Solution:**

The element  $a_{32}$  is 8.

$$M_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

Hence, the minor of  $a_{32}$  is  $\det(M_{32}) = 1(6) - 4(3) = 6 - 12 = -6$ .

The cofactor of  $a_{32}$  is  $(-1)^{3+2} \cdot 6 = -6$ .

## Laplace expansion for determinant

The **determinant** of matrix  $A = [a_{ij}]$  is equal to the sum of the products obtained by multiplying the elements of any row (column) by their respective cofactors:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{j=1}^n a_{ij}A_{ij} \rightarrow \text{row}$$

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{i=1}^n a_{ij}A_{ij} \rightarrow \text{column}$$

The formula is called the **Laplace expansions** of the determinant of  $A$  by the  $i$ -th row and  $j$ -th column.



# Evaluation of determinants

**Algorithm:** (Reduction of the order of a determinant)

**Input:** A non-zero  $n$ -square matrix  $A = [a_{ij}]$  with  $n > 1$

1. Choose an element  $a_{ij} = 1$ , or if there is no any,  $a_{ij} \neq 0$ ;
2. Using  $a_{ij}$  as a pivot, apply elementary row (column) operations to put 0s in all the other positions in the column (row) containing  $a_{ij}$ ;
3. Expand the determinant by the column (row) containing  $a_{ij}$ .

**Remark.**

- The algorithm is usually used for the case  $n \geq 4$ .
- One can implement the Gaussian elimination to transform the matrix into an *upper-triangular* matrix, then compute the determinant as the product of its diagonal. But we need to keep track of the elementary operations performed (as each of them would change the sign of the determinant).

## Example: *computing determinant using cofactors*

Use the algorithm to compute the determinant of:

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{bmatrix}$$

## Example: *computing determinant using cofactors*

Use  $a_{23} = 1$  as a pivot, and apply elementary row operation, then expand the determinant

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix}$$

Hence,

$$\begin{aligned} |A| &= (-1)^{2+3} \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 7 & 0 & 9 \\ -5 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} \\ &= -(-1)^{3+2} \begin{vmatrix} 7 & 9 \\ -5 & -1 \end{vmatrix} \\ &= -7 + 45 = 38 \end{aligned}$$

## Review on the determinants of $(2 \times 2)$ and $(3 \times 3)$ matrices

Let us derive the formula of determinant of  $(2 \times 2)$ -matrices using the algorithm.

$$\text{Given } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} \end{vmatrix}$$

$$\text{Hence, } A = a_{11} \left( a_{22} - \frac{a_{21}a_{12}}{a_{11}} \right) = a_{11} \left( \frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}} \right)$$

# Review on the determinants of $(2 \times 2)$ and $(3 \times 3)$ matrices

## Exercise

Try to derive the formula for the following  $(3 \times 3)$ -matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{32} \end{bmatrix}$$

# Part 7: Applications to linear equations: *Cramer's Rule*

## Cramer's rule

Given a system of linear equations:  $AX = B$ , with  $A = [a_{ij}]$  is the square matrix and  $B = [b_i]$  is the column vector.

Let  $A_i$ : the matrix obtained from  $A$  by replacing the  $i$ -th column of  $A$  by the column vector of  $B$ .

Let:

$$D = \det(A), \quad N_1 = \det(A_1), \quad N_2 = \det(A_2), \quad \dots, \quad N_n = \det(A_n)$$

### Theorem (Cramer's rule)

*The (square) system  $AX = B$  has a solution iff  $D \neq 0$ , and it is given by:*

$$x_1 = \frac{N_1}{D}, \quad x_2 = \frac{N_2}{D}, \quad \dots, \quad x_n = \frac{N_n}{D}$$

**Q:** Give an argument explaining why the theorem holds!

# Notes on the Cramer's rule

- The system must be *square* (it has the same number of equations and variables);
- The solution exists only if  $D \neq 0$ ;
- If  $D = 0$ , it does not tell us whether a solution exists.

For a **square homogeneous system**:

## Theorem

*A square homogeneous system  $AX = 0$  has a nonzero solution if and only if  $D = |A| = 0$ .*



## Example

Apply Cramer's rule to solve the following system:

$$\begin{cases} x + y + z = 5 \\ x - 2y - 3z = -1 \\ 2x + y - z = 3 \end{cases}$$

**Solution:** The coefficient matrix:  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix}$  has determinant

$$D = 2 - 6 + 1 + 4 + 3 + 1 = 5.$$

Since  $D \neq 0$ , the system has a unique solution. Furthermore:

$$N_x = \begin{vmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{vmatrix}, \quad N_y = \begin{vmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{vmatrix}, \quad N_z = \begin{vmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix}$$

This gives:  $N_x = 20$ ,  $N_y = -10$ , and  $N_z = 15$ .

Hence,  $x = \frac{20}{5} = 4$ ,  $y = \frac{-10}{5} = -2$ , and  $z = \frac{15}{5} = 3$ .

# Part 8: Block matrices and determinants

# Block matrices and determinants

## Theorem

Suppose  $M$  is an *upper (lower) triangular block matrix* with the diagonal blocks:

$$A_1, A_2, \dots, A_n$$

Then:

$$\det(M) = \det(A_1) \det(A_2) \cdots \det(A_n)$$

## Example

Given

$$M = \left( \begin{array}{cc|ccc} 2 & 3 & 4 & 7 & 8 \\ -1 & 5 & 3 & 2 & 1 \\ \hline 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 3 & -1 & 4 \\ 0 & 0 & 5 & 2 & 6 \end{array} \right)$$

Evaluate the determinant of each diagonal block:

$$\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} = 13 \qquad \begin{vmatrix} 2 & 1 & 5 \\ 3 & -1 & 4 \\ 5 & 2 & 6 \end{vmatrix} = 29$$

Then  $|M| = 13 \cdot 29 = 377$ .

**Remark.** Suppose  $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$  where  $A, B, C, D$  are square matrices.

Then **it is not generally true** that  $|M| = |A||D| - |B||C|$ .

# Exercise

*Exercises will be given during the lecture...*