

# Linear Algebra

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## 4.2 - Gauss-Jordan Elimination

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# Learning objectives

After this lecture, you should be able to:

- apply the Gauss-Jordan elimination algorithm to solve a system of linear equations.

# Introduction



Carl Friedrich Gauss  
(*German mathematician*)



Wilhelm Jordan  
(*German mathematician*)

# Introduction

This is a development of the Gaussian-Elimination method.

- The ERO is implemented on the augmented matrix, so that a *reduced echelon matrix* is obtained.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & * \end{bmatrix}$$

- The difference with the Gaussian method is that, here **backward substitution is not needed** to obtain the variables values.
- The value of each variable can be derived **directly from the augmented matrix**.

# Steps in Gauss-Jordan method

## 1. Forward phase (Gauss elimination phase)

Under the main diagonal of 1's should be 0.

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{ERO} \begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

## 2. Backward phase

Above the main diagonal of 1's should be 0.

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R1 - (3/2)R2} \begin{bmatrix} 1 & 0 & -5/4 & -11/4 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} R1 + (5/4)R3 \\ R2 - (1/2)R3 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The last matrix is a **reduced row echelon form**.

We can derive the solution directly:  $x_1 = 1$ ,  $x_2 = 2$ ,  $x_3 = 3$ .

## Example 1

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 = -1 \\ 2x_1 + x_2 - 2x_3 - 2x_4 = -2 \\ -x_1 + 2x_2 - 4x_3 + x_4 = 1 \\ 3x_1 \qquad \qquad - 3x_4 = -3 \end{cases}$$

**Solution:**

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \begin{array}{l} R2 - 2R1 \\ R4 - 3R1 \\ R3 + R1 \\ \sim \end{array} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2/3} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \begin{array}{l} R4 - 3R2 \\ R3 - R2 \\ \sim \end{array} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1 + R2} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding equations are:

$$x_1 - x_4 = -1$$

$$x_2 - 2x_3 = 0$$

## Example 1 (*cont.*)

The last augmented matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution can be obtained by solving the system:

$$x_1 - x_4 = -1$$

$$x_2 - 2x_3 = 0$$

From the 2nd eq, we obtain:  $x_2 = 2x_3$

From the 1st eq, we obtain:  $x_1 = x_4 - 1$

Let  $x_3 = r$  and  $x_4 = s$  with  $r, s \in \mathbb{R}$ .

Then the solution of the system is:

$$x_1 = s - 1, \quad x_2 = 2r, \quad x_3 = r, \quad x_4 = s$$

## Example 2

Solve the following system using Gauss-Jordan method

$$\begin{cases} -2x_3 + 7x_5 = 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 = 28 \\ 2x_1 + 4x_2 - 5x_3 + 8x_4 - 5x_5 = -1 \end{cases}$$

**Solution:**

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \begin{matrix} R1 \leftrightarrow R2 \\ \sim \end{matrix} \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \begin{matrix} R1/2 \\ \sim \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \begin{matrix} R3 - 2R1 \\ \sim \end{matrix} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \begin{matrix} R2/(-2) \\ \sim \end{matrix}$$



## Example 2 (cont.)

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \xrightarrow{R3 - 5R2} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \xrightarrow{R3/(1/2)}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\substack{R1 - 6R3 \\ R2 + 7/2R3}} \begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{R1 + 5R2}$$

From the last augmented matrix, can be derived:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{cases} x_1 + 2x_2 + 3x_4 = 7 \\ x_3 = 1 \\ x_5 = 2 \end{cases}$$

Let  $x_2 = s$  and  $x_4 = t$ , the solution of the system:

$$x_1 = 7 - 2s - 3t, \quad x_2 = s, \quad x_3 = 1, \quad x_4 = t, \quad x_5 = 2, \quad s, t \in \mathbb{R}$$

## Example 3

Solve the following system using Gauss-Jordan method

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$$

The augmented matrix:

$$\left[ \begin{array}{cccccc} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array} \right]$$

which can be reduced into row-echelon form:

$$\left[ \begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

## Example 3 (*cont.*)

The corresponding linear equations system is:

$$\begin{cases} x_1 + x_2 & + x_5 = 0 \\ & x_3 + x_5 = 0 \\ & & x_4 = 0 \end{cases}$$

Solving the *leading variables*, we get:

$$x_1 = -x_2 - x_5$$

$$x_3 = -x_5$$

$$x_4 = 0$$

The general solution is:

$$x_1 = -s - t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = 0, \quad x_5 = t, \quad \text{with } s, t \in \mathbb{R}$$

## Analysis of Example 3

What can you observe from the above linear system?

$$\begin{cases} 2x_1 + 2x_2 - x_3 & + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 & - x_5 = 0 \\ & x_3 + x_4 + x_5 = 0 \end{cases}$$

# Homogeneous Linear System

# Homogeneous Linear System

Recall that the following system is called **homogeneous linear system**.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$

.....

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

The solution always has a solution, namely:

$$x_1 = 0, x_2 = 0, \dots, x_n = 0$$

this solution is called **trivial solution**.

If a solution other than  $x_1 = 0, x_2 = 0, \dots, x_n = 0$  exists, then it is called **non-trivial solution**.

# Homogeneous Linear System

## Example

From Example 3 of the previous section, we obtain the solution of the given linear system is:

$$x_1 = -s - t, \quad x_2 = s, \quad x_3 = -t, \quad x_4 = 0, \quad x_5 = t, \quad \text{with } s, t \in \mathbb{R}$$

Here, if  $s, t = 0$ , then we get the **trivial solution**, namely:

$$x_1 = 0, \quad x_2 = 0, \quad x_3 = 0, \quad x_4 = 0, \quad x_5 = 0$$

We can set  $s \neq 0$  or  $t \neq 0$  to get **non-trivial solutions**.

## Example

Solve the following homogeneous linear system by Gauss-Jordan elimination:

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R2} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \xrightarrow{\begin{matrix} R3 - 2R1 \\ R4 + 2R1 \end{matrix}} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix}$$

$$\xrightarrow{R2/2} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \xrightarrow{\begin{matrix} R3 - 3R2 \\ R4 - R2 \end{matrix}} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \xrightarrow{\begin{matrix} R1 + 3R3 \\ R2 - 2R3 \\ R4 + 10R3 \end{matrix}} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



# Analysis

Why the following matrix has a reduced-row echelon form?

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Analysis of homogeneous system

*When does a homogeneous linear system has an infinitely many solutions?*

# Analysis of homogeneous system

*When does a homogeneous linear system has an infinitely many solutions?*

*How many free variables do exist in a homogeneous linear system (relate it to the value of  $n$  and  $r$ )?*

# Analysis of homogeneous system

## Theorem

*If a homogeneous linear system has  $n$  unknowns, and if the reduced row echelon form of its augmented matrix has  $r$  nonzero rows, then the system has  $n - r$  free variables.*

## Theorem

*A homogeneous linear system with more unknowns than equations has infinitely many solutions.*