Linear Algebra

[KOMS119602] - 2022/2023

14.1 - Diagonalization

Dewi Sintiari

Computer Science Study Program
Universitas Pendidikan Ganesha

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Learning objectives

After this lecture, you should be able to:

- verify whether a matrix is orthogonal or not;
- perfom orthogonal diagonalization on a matrix.

Orthogonal matrix

Orthogonal matrix

- The really nice bases of $\mathbb{R}^n$ are the orthogonal bases, so a natural question is: **which $n \times n$ matrices have an orthogonal basis of eigenvectors?**

Orthogonal matrix

A square matrix A is said to be orthogonal if:

$$A^{-1} = A^T$$

or, equivalently if $AA^T = A^T A = I$.

Example

The following matrix is orthogonal.

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

Task: Prove it!

Example solution

We show that $AA^T = I$ (*orthogonality property*).

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix} \begin{bmatrix} \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$$

$$= \frac{1}{49} \begin{bmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{bmatrix} \begin{bmatrix} 3 & -6 & 2 \\ 2 & 3 & 6 \\ 6 & 2 & -3 \end{bmatrix}$$

$$= \frac{1}{49} \begin{bmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of orthogonal matrix

Let A be an $n \times n$ matrix. The following are equivalent.

1. A is orthogonal.
2. The row vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.
3. The column vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product.

A set of matrix forms an orthonormal set if the vectors are **pairwise orthogonal**, and the magnitude of every vector is 1.

Why is orthogonal matrix important?

- They are involved in some of the most important decompositions in numerical Linear Algebra, such as: **QR-decomposition**, **Singular Value Decomposition (SVD)**, etc.

Exercise: Give another example of importance of orthogonal matrix!

Exercise

Recall the rotation matrix transformation in $\mathbb{R}^2$.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Is matrix A orthogonal?

What about the following matrices?

1. Reflection matrix in $\mathbb{R}^2$ and $\mathbb{R}^3$?
2. Orthogonal projection on $\mathbb{R}^2$ and $\mathbb{R}^3$?
3. Rotation on $\mathbb{R}^3$?

Exercise (solution for rotation matrix)

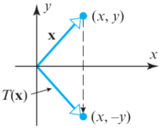
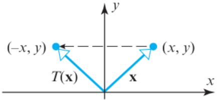
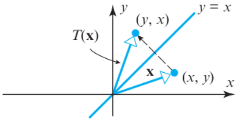
$$\det(A) = \cos^2(\theta) + \sin^2(\theta) = 1$$

Hence:

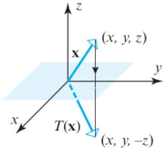
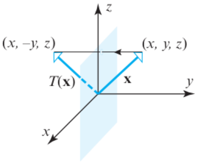
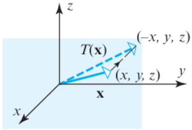
$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = A^T$$

So, the rotation matrix in $\mathbb{R}^2$ is an orthogonal matrix.

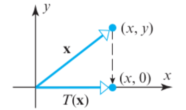
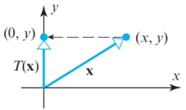
Exercise 1: Reflection operators on $\mathbb{R}^3$

Operator	Illustration	Images of $\mathbf{e}_1$ and $\mathbf{e}_2$	Standard Matrix
Reflection about the x-axis $T(x, y) = (x, -y)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, -1)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the y-axis $T(x, y) = (-x, y)$		$T(\mathbf{e}_1) = T(1, 0) = (-1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the line $y = x$ $T(x, y) = (y, x)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 1)$ $T(\mathbf{e}_2) = T(0, 1) = (1, 0)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

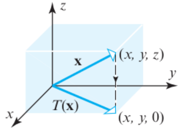
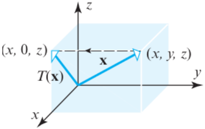
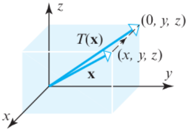
Exercise 2: Reflection operators on $\mathbb{R}^3$

Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Reflection about the xy-plane $T(x, y, z) = (x, y, -z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, -1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane $T(x, y, z) = (x, -y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, -1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-plane $T(x, y, z) = (-x, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (-1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

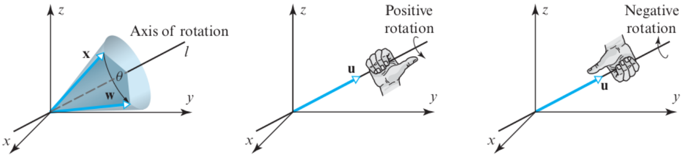
Exercise 3: Projection operators on $\mathbb{R}^3$

Operator	Illustration	Images of e_1 and e_2	Standard Matrix
Orthogonal projection onto the x-axis $T(x, y) = (x, 0)$		$T(\mathbf{e}_1) = T(1, 0) = (1, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 0)$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Orthogonal projection onto the y-axis $T(x, y) = (0, y)$		$T(\mathbf{e}_1) = T(1, 0) = (0, 0)$ $T(\mathbf{e}_2) = T(0, 1) = (0, 1)$	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Exercise 4: Projection operators on $\mathbb{R}^3$

Operator	**Illustration**	**Images of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$**	**Standard Matrix**
Orthogonal projection onto the xy-plane $T(x, y, z) = (x, y, 0)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 0)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Orthogonal projection onto the xz-plane $T(x, y, z) = (x, 0, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (1, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 0, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Orthogonal projection onto the yz-plane $T(x, y, z) = (0, y, z)$		$T(\mathbf{e}_1) = T(1, 0, 0) = (0, 0, 0)$ $T(\mathbf{e}_2) = T(0, 1, 0) = (0, 1, 0)$ $T(\mathbf{e}_3) = T(0, 0, 1) = (0, 0, 1)$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Exercise 5: Rotations in $\mathbb{R}^3$



(a) Angle of rotation (b) Right-hand rule

Orthogonal diagonalization

What is *orthogonal diagonalization*?

Let A and B be square matrices. B is said to be orthogonally similar to A, if there is an orthogonal matrix P, s.t.:

$$B = P^T A P$$

Remark. Conversely, A is also orthogonally similar to B. Can you explain why?

What is *orthogonal diagonalization*?

Let A and B be square matrices. B is said to be orthogonally similar to A, if there is an orthogonal matrix P, s.t.:

$$B = P^T AP$$

Remark. Conversely, A is also orthogonally similar to B. Can you explain why?

Proof. Take $Q = P^T$. Then:

$$Q^T BQ = PBP^T = A$$

(because $B = P^T AP \Rightarrow PBP^T = P(P^T AP)P^T = IAI = A$, since $P^T = P^{-1}$)

Orthogonal diagonalization

If a square matrix A is orthogonally similar to some diagonal matrix D, i.e.

$$P^T AP = D$$

then we say that A is orthogonally diagonalizable and that P orthogonally diagonalizes A.

Why do we care about orthogonal diagonalization?

What type of matrix that is diagonalizable?

Lemma

A square matrix is orthogonally diagonalizable if and only if it is **symmetric**.

Proof.

□

Algorithm for orthogonally diagonalization

Let A be an $n \times n$ symmetric matrix.

- **Step 1.** Find a basis for each eigenspace of A.
- **Step 2.** Apply the Gram-Schmidt process to each of these bases to obtain an orthonormal basis for each eigenspace.
- **Step 3.** Form the matrix P whose columns are the vectors constructed in Step 2.

Matrix P is a matrix that will orthogonally diagonalize A, i.e.

$$D = P^T AP \quad \text{is a diagonal matrix}$$

Exercises

Orthogonally diagonalize the following matrices:

- $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$
- $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Solution of exercises

Solution can be found in
https://psu.pb.unizin.org/psumath220lin/chapter/section-5-2-orthogonal-diagonalization/

