13 - Backtracking

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Design and Analysis of Algorithm (2021/2022)

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Principal of Backtracking

- The *exhaustive-search* technique suggests generating all candidate solutions and then identifying the one (or the ones) with a desired property.
- Backtracking algorithm improves exhaustive search.
- In exhaustive search, all possible solutions are explored and evaluated one-by-one
- In backtracking, we do not examine all possibilities, only the possibilities that lead to the solution. Other nodes that do not lead to the solution are pruned.

Central idea:
To cut off a branch of the problems state-space tree, as soon as we can deduce that it cannot lead to a solution.

Principal of Backtracking

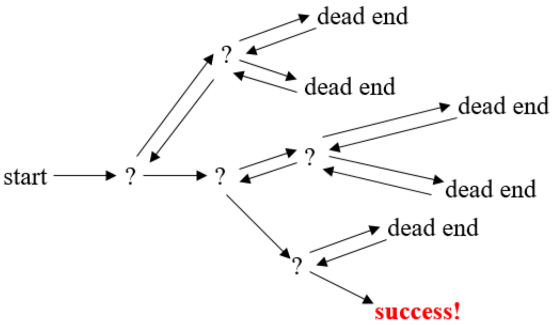


Figure: Illustration of backtracking (sumber: `https://miro.medium.com/`)

Representation of solution

- Representation: an output can be thought of as n-tuple $(x_1, x_2, \ldots, x_n)$ where each coordinate x_i is an element of some finite linearly ordered set S_i.
- Tuples: all solution tuples can be of the same length (the n-queens and the Hamiltonian circuit problem) and of different lengths (the Subset-sum problem).

Backtracking in DFS

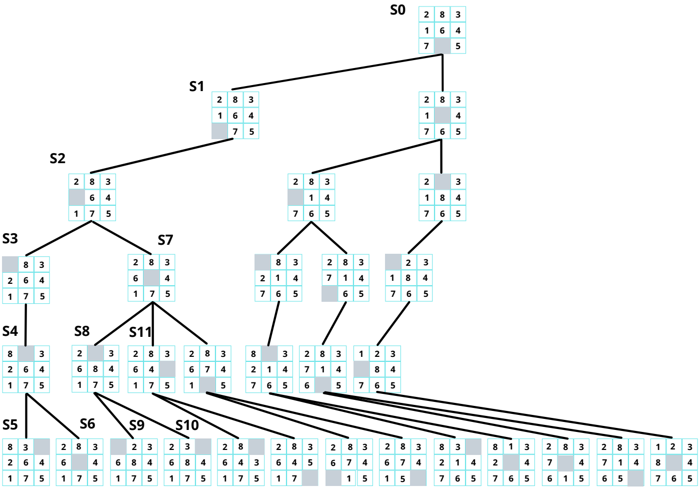
Backtracking in DFS is used in solution-searching problems that have many possibilities of solution.

The solution is obtained by looking in a depth-first approach

- You do not have enough information to know the next step.
- Each decision leads you to several/many new choices.
- Several sequence of choices may be the problem's solution.

In DFS, backtracking is used as a methodological way to try several sequences of decision.

Example of backtracking in DFS



State-space tree

State-space tree (1)

Backtracking can be seen as searching in a tree from the root to the leaves (solution node).

State-space tree

It is a tree representing all the possible states (solution or non-solution) of the problem from the root as an initial state to the leaf as a terminal state.

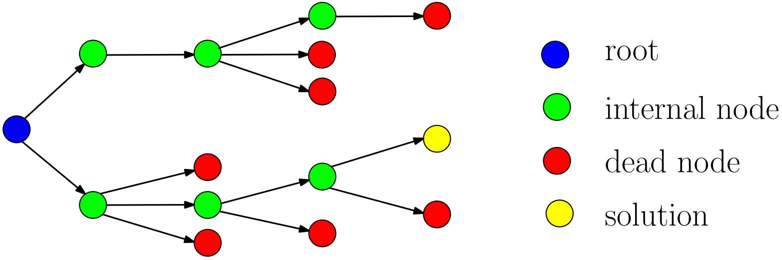
A backtracking algorithm generates, explicitly or implicitly, a **state-space tree**:

- its nodes represent partially constructed tuples with the first i coordinates defined by the earlier actions of the algorithm;
- if such a tuple $(x_1, x_2, \ldots, x_i)$ is not a solution, the algorithm finds the next element in S_{i+1} that is consistent with the values of $(x_1, x_2, \ldots, x_i)$ and the problems constraints, and adds it to the tuple as its $(i+1)$st coordinate;
- if such an element does not exist, the algorithm backtracks to consider the next value of x_i, and so on.

State-space tree (2)

- Root represents an initial state before the search begins;
- Internal nodes
  - the nodes of the first level in the tree represent the choices made for the first component of a solution;
  - the nodes of the second level represent the choices for the second component;
  - and so on...;
- Leaves represent either non-promising dead ends or complete solutions found by the algorithm.

State-space tree (3)



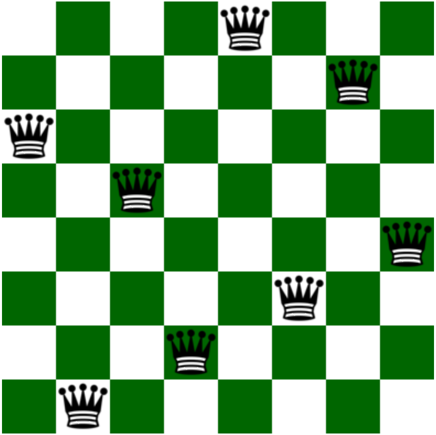
Types of nodes in the state-space tree

- Promising node: corresponds to a partially constructed solution that may still lead to a complete solution;
- Non-promising node: dead node

State-space tree (4)

- The solution is searched by generating the state nodes, so that it produces paths from the root to the leaves;
- To generate the nodes, the DFS rule is followed;
- The generated nodes are called live node;
- The live node being expanded is called expand-node;
- Each time the expand-node is expanded, the generated path gets longer;
- Function that is used to "kill" an expand-node is called bounding function;
- When a node is killed, then automatically all its children nodes are pruned;
- If the paths-generation ends up with dead node, the searching is backtrack to the parents nodes;
- These parents nodes become the new expand-nodes;
- The searching is stopped if we find a solution.

n-Queens problem

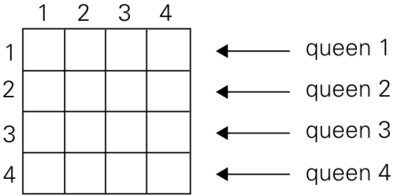


n-Queens problem

Problem

Place n queens on an $n \times n$ chessboard, so that no two queens attack each other (by being in the same row, in the same column, or in the same diagonal).

- For $n = 1$, the problem has a trivial solution
- For $n = 2, 3$, the problem has no solution
- What if $n = 4$?



Algorithm

START

1. begin from the leftmost column
2. if all the queens are placed, return true/ print configuration
3. check for all rows in the current column
    1. if queen placed safely, mark row and column; and recursively check if we approach in the current configuration, do we obtain a solution or not
    2. if placing yields a solution, return true
    3. if placing does not yield a solution, unmark and try other rows
4. if all rows tried and solution not obtained, return false and backtrack

END

n-Queens problem

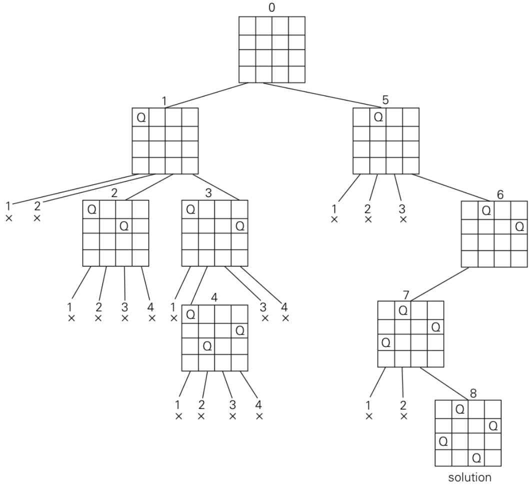


Figure: State-space tree of solving the four-queens problem by backtracking; $\times$ denotes an unsuccessful attempt to place a queen in the indicated column. The numbers above the nodes indicate the order in which the nodes are generated.

n-Queens problem

1. We start with the empty board and then place queen 1 in the first possible position of its row, which is in column 1 of row 1.
2. Then we place queen 2, after trying unsuccessfully columns 1 and 2, in the first acceptable position for it, which is square $(2, 3)$, the square in row 2 and column 3.
3. This proves to be a dead end because there is no acceptable position for queen 3.
4. So, the algorithm backtracks and puts queen 2 in the next possible position at $(2, 4)$.
5. Then queen 3 is placed at $(3, 2)$, which proves to be another dead end.
6. The algorithm then backtracks all the way to queen 1 and moves it to $(1, 2)$.
7. Queen 2 then goes to $(2, 4)$, queen 3 to $(3, 1)$, and queen 4 to $(4, 3)$, which is a solution to the problem.

Other problems

Hamiltonian circuit problem

Problem

Given a connected graph G, find a Hamiltonian circuit in G. (Recall that a Hamiltonian circuit is a circuit that visits all vertices of G exactly once.)

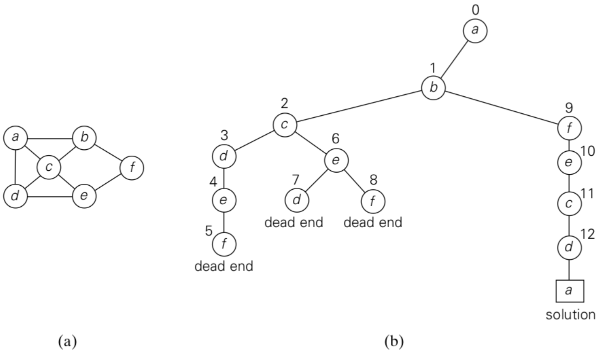


Figure: (a) Graph. (b) State-space tree for finding a Hamiltonian circuit. The numbers above the nodes of the tree indicate the order in which the nodes are generated.

Subset-sum problem (1)

Problem

Find a subset of a given set $A = \{a_1, \ldots, a_n\}$ *of* n *positive integers whose sum is equal to a given positive integer* d*.*

Example 1: Given $A = \{1, 2, 5, 6, 8\}$, $d = 9$, the solution are: $\{1, 2, 6\}$ and $\{1, 8\}$.

Example 2: Given $A = \{3, 5, 6, 7\}$, $d = 15$, the solution are: $\{3, 5, 7\}$.

Subset-sum problem (2)

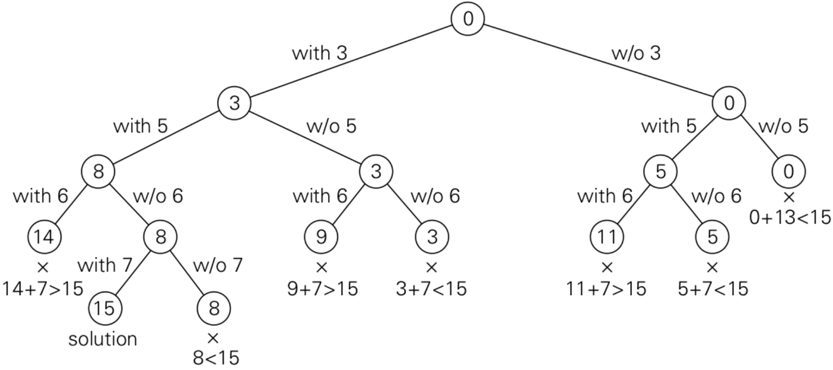


Figure: Complete state-space tree of the backtracking algorithm applied to the instance $A = \{3, 5, 6, 7\}$ and $d = 15$ of the Subset-sum problem. The number inside a node is the sum of the elements already included in the subsets represented by the node. The inequality below a leaf indicates the reason for its termination.

Subset-sum problem (1)

A path from the root to a node on the ith level of the tree indicates which of the first i numbers have been included in the subsets represented by that node.

We record the value of s, the sum of these numbers, in the node.

- If s is equal to d, we have a solution to the problem. We can either report this result and stop or,
- If all solutions need to be found, continue by backtracking to the nodes parent.
- If s is not equal to d, we can terminate the node as non-promising if either of the following two inequalities holds:

$$s + a_{i+1} > d \text{ (the sum } s \text{ is too large)}$$

$$s + \sum_{j=i+1}^{n} a_j < d \text{ (the sum } s \text{ is too small)}$$

Backtracking framework

Backtracking algorithm framework

Algorithm 1 Backtracking

procedure BACKTRACK($X[1..i]$)
    input: $X[1..i]$: the first i promising components of a solution
    output: all the tuples representing the problem's solution
    if $X[1..i]$ is a solution **then**
        write $(X[1..i])$
    else
        for each $x \in S_{i+1}$ consistent with $X[1..i]$ and the constraints **do**
            $X[i+1] \leftarrow x$
            BACKTRACK($X[1..i+1]$)
        end for
    end if
end procedure

Time complexity

Backtracking is basically an exhaustive search performed over the search space. So the time complexity of a backtracking algorithm is **defined by the size of the search space**.

For example, in the n-queens problem and Hamiltonian problem, the size of the search space is about $\mathcal{O}(n!)$.

Intuitively, the first queen has n placements, the second queen must not be in the same column as the first, so the second queen has $n - 1$ possibilities, and so on, with a time complexity of $\mathcal{O}(n!)$.

Advantages & drawbacks

Advantages

- Typically applied to difficult combinatorial problems for which no efficient algorithms for finding exact solutions possibly exist.
- Unlike the exhaustive-search approach,backtracking at least holds a hope for solving some instances of nontrivial sizes in an acceptable amount of time (especially for optimization problems).
- Even if backtracking does not eliminate any elements of a problems state space and ends up generating all its elements, it provides a specific technique for doing so.

Drawbacks

- Backtracking is **not** a very efficient technique (even though it was succeeded to use in the previous problems).
- In the worst case, it may have to generate all possible candidates in an exponentially (or faster) growing state space of the problem.