Linear Algebra

[KOMS120301] - 2023/2024

6.1 - Inverses of matrices

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Week 6 (October 2023)

Learning objectives

After this lecture, you should be able to:

1. investigate if a matrix inverse exists;
2. compute the inverse of a *small size* matrix (if exists);
3. compute the inverse of an $n \times n$ matrix (if exists);
4. explain the concepts of *minor*, *cofactor*, *adjoint*;
5. explain the properties of matrix inverse;
6. analyze if a matrix is orthogonal;
7. analyze if a set of vectors is orthonormal.

Part 1: Inverse of matrices

Inverse

A square matrix A is said to be invertible or nonsingular if $\exists B$ s.t.:

$$AB = BA = I \qquad \text{where } I \text{ is the identity matrix}$$

Note: Such a matrix B is unique, and it is called the inverse of A, and is denoted by A^{-1}. The relation of A and B is symmetric:

If B is the inverse of A, then A is the inverse of B, i.e.

$$(A^{-1})^{-1} = A$$

Example

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ Then

$$AB = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Why do we need to find **inverse** of a matrix?

1. 'Primarily, "division" does not exist for matrices, instead, we do "inverse".

   Given a matrix A and B such that $B = AX$.
   How do we find X? $\Rightarrow X = BA^{-1}$

2. Applications:
   - solving a system of linear equations;
   - used to encrypt/decrypt message codes;
   - etc.

How to compute the inverse of 2×2 matrices?

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *what is* A^{-1}*?*

Let $A^{-1} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$. We have:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} ax_1 + by_1 & ax_2 + by_2 \\ cx_1 + dy_1 & cx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We solve the linear system:

$$\begin{cases} ax_1 + by_1 & = 1 \\ cx_1 + dy_1 & = 0 \end{cases} \quad \text{and} \quad \begin{cases} ax_2 + by_2 & = 0 \\ cx_2 + dy_2 & = 1 \end{cases}$$

Inverse of 2×2 matrices

It gives:

$$x_1 = \frac{d}{ad - bc}, \quad y_1 = \frac{-c}{ad - bc}, \quad x_2 = \frac{-b}{ad - bc}, \quad y_2 = \frac{a}{ad - bc}$$

Note that $ad - bc = |A|$ (the *determinant* of A).

When $|A| \neq 0$, the values x_1, y_1, x_2, and y_2 exist.

Hence,

$$A^{-1} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} d/|A| & -b/|A| \\ -c/|A| & a/|A| \end{bmatrix} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of 2×2 matrices

Conclusion:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

When $|A| \neq 0$, the inverse of a 2×2 matrix A may be obtained from A as follows:

1. Interchange the two elements on the diagonal (a and d);
2. Take the negatives of the other two elements (b and c);
3. Multiply the resulting matrix by $\frac{1}{|A|}$ or, equivalently, divide each element by $|A|$.

Note: If $|A| = 0$, then A is not invertible.

Example

Find the inverse of $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$|A| = 2(5) - 3(4) = 10 - 12 = -2$$

Since $|A| \neq 0$, then A is invertible.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

Furthermore, $|B| = 1(6) - 3(2) = 0$, so B is not invertible.

Part 2: Computing inverse from adjoint

Inverse of $n \times n$ matrices

Note:

If A is an $n \times n$ matrices, A^{-1} can be obtained as above, by finding the solution of the $n \times n$ linear system equations.

This is not so practical to be solved using the substitution/elimination method. A method will be discussed later.

Review on minors and cofactors

Let $A = [a_{ij}]$ be an n-square matrix.

Define M_{ij} as the $(n-1)$-square matrix obtained from A by deleting the i-th row and the j-th column of A.

The minor of the element a_{ij} of A is defined as:

$$\text{minor}(A) = \det(M_{ij})$$

The cofactor of a_{ij} is defined as the signed minor of a_{ij}, and denoted by:

$$C_{ij} = (-1)^{i+j} \, |M_{ij}|$$

Adjoint

We can form a matrix of cofactors

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

where C_{ij} is the cofactor of a_{ij}.

The adjoint of matrix A is defined as:

$$\text{adj}(A) = C^T$$

Example of adjoint

Given matrix:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The cofactors of A are:

- $C_{11} = 12$
- $C_{12} = 6$
- $C_{13} = -16$
- $C_{21} = 4$
- $C_{22} = 2$
- $C_{23} = 16$
- $C_{31} = 12$
- $C_{13} = -10$
- $C_{33} = 16$

The matrix of cofactors and the adjoint of A are:

$$C = \begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix} \qquad \text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Matrix inverse from adjoint

Theorem

Let A be an **invertible** *matrix. Then:*

$$A^{-1} = \frac{1}{\det(A)}\ adj(A)$$

Proof can be read on the Howard Anton book, page 134.

Algorithm for inverse computation using adjoint

Suppose that $A = [a_{ij}]$ is a matrix of size $n \times n$. We want to compute A^{-1}

1. For each element a_{ij}, find matrix M_{ij}.
2. Compute the minor of M_{ij}, namely $\text{minor}(a_{ij}) = |M_{ij}|$.
3. Compute the cofactor of a_{ij}, namely $C_{ij} = (-1)^{i+j} \cdot |M_{ij}|$.
4. Build the cofactor matrix $C = [C_{ij}]$.
5. Find the adjoint of A, namely $Adj(A) = C^T$.
6. Compute the inverse of A, namely:

$$A^{-1} = \frac{1}{\det(A)} \cdot Adj(A)$$

Example

From the *previous example*, we have:

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix} \qquad \text{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

$$\det(A) = 0 + 12 + 4 - (-12 - 36 + 0) = 16 - (-48) = 64$$

Hence,

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & \frac{-10}{64} \\ \frac{-16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

Part 3: Properties of matrix inverse

Basic properties of matrix inverse

Let A be an **invertible** matrix. The followings hold.

1. $(A^{-1})^{-1} = A$
2. $(kA)^{-1} = k^{-1}A^{-1}$ for a scalar $k \neq 0 \in \mathbb{R}$
3. $(A^T)^{-1} = (A^{-1})^T$
4. $\det(A^{-1}) = (\det(A))^{-1}$

Exercises:

Prove the properties of matrix inverse.

Give an example for each property to check the correctness of the theorem.

Basic properties of matrix inverse

Theorem

If A and B are invertible, then AB is invertible.

Proof.

Consider $B^{-1}A^{-1}$. Then:

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

Hence, AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$. □

Generalization:

If $A_1, A_2, \ldots, A_k$ are invertible matrices, then:

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}$$

Exercise

will be given during the lecture

Numbers 4, 5, 6, page 76 Howard Anton Reference Book

Part 4: Orthogonal matrices

Orthogonal matrices

A matrix is called orthogonal if $A^T = A^{-1}$, i.e., $AA^T = A^TA = I$ (the identity matrix).

Note: A is orthogonal *only if* A is square and invertible matrix.

Example

Let $A = \begin{bmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{4}{9} & -\frac{4}{9} & -\frac{7}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \end{bmatrix}$

Is A orthogonal? What is the result of AA^T?

Orthonormality

Vectors $u_1, u_2, \ldots, u_m$ in $\mathbb{R}^n$ are said to form an orthonormal set of vectors if the vectors are unit vectors and are orthogonal to each other; i.e.,

$$u_i \cdot u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Theorem

Let A be a real matrix. Then the following are equivalent:

- *A is orthogonal.*
- *The rows of A form an orthonormal set.*
- *The columns of A form an orthonormal set.*

to be continued...