Linear Algebra
[KOMS120301] - 2023/2024

5.2 - Determinants of Matrices

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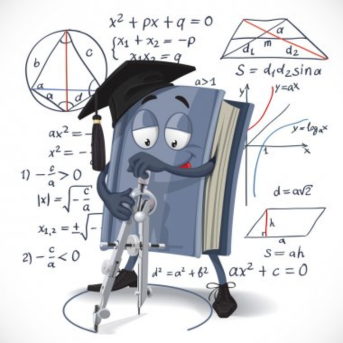
Learning objectives

After this lecture, you should be able to:

1. implement the properties of determinant in problem solving;
2. compute determinant of matrix using cofactor expansion;
3. solve a system of linear equations using the Cramer's rule;
4. explain the procedure of computing the determinant of a diagonal block matrix.

Good math skills are developed by doing lots of problems.



Part 5: Properties of determinant

Determinant of the matrix transpose

Theorem

$$\det(A^T) = \det(A)$$

Q: Can you explain why? Check it for the 2×2 matrix and the 3×3 matrix.

Implication:

> *Any theorem about the determinant of a matrix A that concerns the rows of A will have an analogous theorem concerning the columns of A.*

Basic properties of determinant

Theorem (Determinants related to the *shape* of the matrix)

Let A be a square matrix.

1. *If A has a row (column) of zeros, then $|A| = 0$.*
2. *If A has two identical rows (columns), then $|A| = 0$.*
3. *If A is triangular (i.e., A has zeros above or below the diagonal), then $|A| =$ product of the diagonal elements:*

$$|A| = \prod_{i=1}^{n} a_{ii}$$

Particularly, for the identity matrix I, we have $|I| = 1$.

Q: Give an argument explaining why those properties hold!

Elementary operations and determinant

Theorem (Determinants & Elementary Row Operations)

Suppose B is obtained from A by an elementary row (column) operation.

1. *If two rows (columns) of A were interchanged, then $|B| = -|A|$.*
2. *If a row (column) of A were multiplied by a scalar k, then $|B| = k|A|$.*
3. *If a multiple of a row (column) of A were added to another row (column) of A, then $|B| = |A|$.*

Q: Give an argument explaining why those properties hold!

Determinant of matrix product

Theorem (Determinant of matrix product)

Given two square matrices A and B of the same order. Then:

$$\det(AB) = \det(A) \cdot \det(B)$$

Proof: `https://proofwiki.org/wiki/Determinant_of_Matrix_Product`

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An elementary matrix E_n is a matrix which differs from the identity matrix I_n by one single elementary row operation.

Corollary

If E is an elementary matrix of size n, and A is an $n \times n$ square matrix. Then $|EA| = |E||A|$.

Exercise

Given:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$$

1. Compute AB
2. Compute $\det(A)$, $\det(B)$, and $\det(AB)$.
3. Is it true that $\det(A) \cdot \det(B) = \det(AB)$?

Exercise

Given:

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}$$

1. Compute AB

$$AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

2. Compute $\det(A)$, $\det(B)$, and $\det(AB)$.

$$\det(A) = 1, \ \det(B) = -23, \ \det(AB) = -23$$

3. Is it true that $\det(A) \cdot \det(B) = \det(AB)$?

Part 6: Computing determinant by cofactor expansion (*an algorithmic approach*)

Minors and cofactors

Let $A = [a_{ij}]$ be an n-square matrix.
Let M_{ij} be the $(n-1)$-square matrix obtained from A by deleting the i-th row and the j-th column of A.

The minor of the element a_{ij} of A is defined as:

$$\text{minor}(A) = \det(M_{ij})$$

The cofactor of a_{ij} is defined as the signed minor of a_{ij}, and denoted by:

$$A_{ij} = (-1)^{i+j}\ |M_{ij}|$$

The pattern of the sign minor of elements in A can be written as:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix}$$

Example: *minors and cofactors*

Given matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the minor and the cofactor of the element a_{32}!

Example: *minors and cofactors*

Given matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Find the minor and the cofactor of the element a_{32}!

Solution:

The element a_{32} is 8.

$$M_{32} = \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix}$$

Hence, the minor of a_{32} is $\det(M_{32}) = 1(6) - 4(3) = 6 - 12 - 6$.

The cofactor of a_{32} is $(-1)^{3+2} \cdot 6 = -6$.

Laplace expansion for determinant

The determinant of matrix $A = [a_{ij}]$ is equal to the sum of the products obtained by multiplying the elements of any row (column) by their respective cofactors:

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} = \sum_{i=1}^{n} a_{ij}A_{ij} \quad \rightarrow row$$

$$|A| = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} = \sum_{j=1}^{n} a_{ij}A_{ij} \quad \rightarrow column$$

The formula is called the Laplace expansions of the determinant of A by the i-th row and j-th column.

Evaluation of determinants

Algorithm: (Reduction of the order of a determinant)

Input: A non-zero n-square matrix $A = [a_{ij}]$ with $n > 1$

1. Choose an element $a_{ij} = 1$, or if there is no any, $a_{ij} \neq 0$;
2. Using a_{ij} as a pivot, apply elementary row (column) operations to put 0s in all the other positions in the column (row) containing a_{ij};
3. Expand the determinant by the column (row) containing a_{ij}.

Remark.

- The algorithm is usually used for the case $n \geq 4$.
- One can implement the Gaussian elimination to transform the matrix into an *upper-triangular* matrix, then compute the determinant as the product of its diagonal. But we need to keep track of the elementary operations performed (as each of them would change the sign of the determinant).

Example: *computing determinant using cofactors*

Use the algorithm to compute the determinant of:

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{bmatrix}$$

Example: *computing determinant using cofactors*

Use $a_{23} = 1$ as a pivot, and apply elementary row operation, then expand the determinant

$$|A| = \begin{vmatrix} 5 & 4 & 2 & 1 \\ 2 & 3 & 1 & -2 \\ -5 & -7 & -3 & 9 \\ 1 & -2 & -1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 0 & 5 \\ 2 & 3 & 1 & -2 \\ 1 & 2 & 0 & 3 \\ 3 & 1 & 0 & 2 \end{vmatrix}$$

Hence,

$$\begin{aligned} |A| &= (-1)^{2+3} \begin{vmatrix} 1 & -2 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix} = - \begin{vmatrix} 7 & 0 & 9 \\ -5 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} \\ &= -(-1)^{3+2} \begin{vmatrix} 7 & 9 \\ -5 & -1 \end{vmatrix} \\ &= -7 + 45 = 38 \end{aligned}$$

Review on the determinants of (2×2) and (3×3) matrices

Let us derive the formula of determinant of (2×2)-matrices using the algorithm.

Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \begin{vmatrix} 1 & \frac{a_{12}}{a_{11}} \\ 0 & a_{22} - \frac{a_{21}a_{12}}{a_{11}} \end{vmatrix}$$

Hence, $A = a_{11}\left(a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right) = a_{11}\left(\frac{a_{11}a_{22} - a_{21}a_{12}}{a_{11}}\right)$

Review on the determinants of (2×2) and (3×3) matrices

Exercise

Try to derive the formula for the following (3×3)-matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{32} \end{bmatrix}$$

Part 7: Applications to linear equations: *Cramer's Rule*

Cramer's rule

Given a system of linear equations: $AX = B$, with $A = [a_{ij}]$ is the square matrix and $B = [b_i]$ is the column vector.

Let A_i: the matrix obtained from A by replacing the i-th column of A by the column vector of B.

Let:

$$D = \det(A), \quad N_1 = \det(A_1), \quad N_2 = \det(A_2), \quad \ldots, \quad N_n = \det(A_n)$$

Theorem (Cramer's rule)

The (square) system $AX = B$ has a solution iff $D \neq 0$, and it is given by:

$$x_1 = \frac{N_1}{D}, \quad x_2 = \frac{N_2}{D}, \quad \ldots, \quad x_n = \frac{N_n}{D}$$

Q: Give an argument explaining why the theorem holds!

Notes on the Cramer's rule

- The system must be *square* (it has the same number of equations and variables);
- The solution exists only if $D \neq 0$;
- If $D = 0$, it does not tell us whether a solution exists.

For a **square homogeneous system**:

Theorem

A square homogeneous system $AX = 0$ has a nonzero solution if and only if $D = |A| = 0$.

Example

Apply Cramer's rule to solve the following system:

$$\begin{cases} x + y + z = 5 \\ x - 2y - 3z = -1 \\ 2x + y - z = 3 \end{cases}$$

Solution: The coefficient matrix: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -3 \\ 2 & 1 & -1 \end{bmatrix}$ has determinant $D = 2 - 6 + 1 + 4 + 3 + 1 = 5$.

Since $D \neq 0$, the system has a unique solution. Furthermore:

$$N_x = \begin{vmatrix} 5 & 1 & 1 \\ -1 & -2 & -3 \\ 3 & 1 & -1 \end{vmatrix}, \quad N_y = \begin{vmatrix} 1 & 5 & 1 \\ 1 & -1 & -3 \\ 2 & 3 & -1 \end{vmatrix}, \quad N_z = \begin{vmatrix} 1 & 1 & 5 \\ 1 & -2 & -1 \\ 2 & 1 & 3 \end{vmatrix}$$

This gives: $N_x = 20$, $N_y = -10$, and $N_z = 15$.
Hence, $x = \frac{20}{5} = 4$, $y = \frac{-10}{5} = -2$, and $x = \frac{15}{5} = 3$.

Part 8: Block matrices and determinants

Block matrices and determinants

Theorem

Suppose M is an upper (lower) triangular block matrix with the diagonal blocks:

$$A_1, A_2, \ldots, A_n$$

Then:

$$\det(M) = \det(A_1)\det(A_2)\cdots\det(A_n)$$

Example

Given

$$M = \left(\begin{array}{cc|ccc} 2 & 3 & 4 & 7 & 8 \\ -1 & 5 & 3 & 2 & 1 \\ \hline 0 & 0 & 2 & 1 & 5 \\ 0 & 0 & 3 & -1 & 4 \\ 0 & 0 & 5 & 2 & 6 \end{array}\right)$$

Evaluate the determinant of each diagonal block:

$$\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} = 13 \qquad \begin{vmatrix} 2 & 1 & 5 \\ 3 & -1 & 4 \\ 5 & 2 & 6 \end{vmatrix} = 29$$

Then $|M| = 13 \cdot 29 = 377$.

Remark. Suppose $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ where A, B, C, D are square matrices.

Then **it is not generally true** that $|M| = |A||D| - |B||C|$.

Exercise

Exercises will be given during the lecture...