Linear Algebra

[KOMS120301] - 2023/2024

5.1 - Determinants of Matrices

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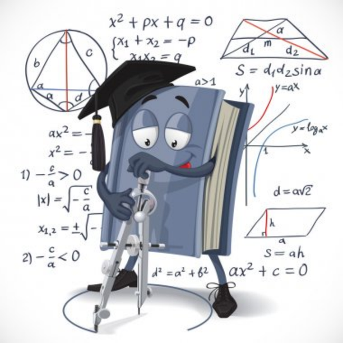
Week 5 (September 2023)

Learning objectives

After this lecture, you should be able to:

1. explain the concept of determinant of a matrix;
2. compute the determinant of (2×2) matrices;
3. compute the determinant of (3×3) matrices;
4. explain the geometric interpretation of determinant of (2×2) matrices;
5. explain the geometric interpretation of determinant of (3×3) matrices;
6. explain the use of determinant in the system of liner equations;
7. use permutation to compute determinants;

Good math skills are developed by doing lots of problems.



Part 1: Formal definition of determinant

Formal definition of determinant matrix

Given a square matrix $A = [a_{ij}]$ of size $n \times n$.

We can assign a *scalar* to matrix A, as a function of the entries of the square matrix. This is called the determinant of matrix A.

The determinant of matrix A is denoted by $|A|$, and often written as:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Determinants of orders 1 and 2

For $n = 1, 2$, the determinants are defined as:

$$|a_{11} = a_{11}| \qquad \text{and} \qquad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example

Find the determinant of the following matrices:

$$\begin{bmatrix} 5 & 3 \\ 2 & 6 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 4 & -5 \\ -6 & 3 \end{bmatrix}$$

Part 2: Determinants of 2×2 matrices

Determinants of 2 × 2 matrices

Given a matrix:

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

In high school, you might have learned that the determinant of the matrix (size 2 × 2) is defined as

$$a_1 b_2 - a_2 b_1$$

and is denoted by:

$$|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Motivating example: *an important application of determinant*

Recall that, given a system of linear equations in two variables:

$$
\begin{aligned}
a_1x + b_1y &= c_1 \\
a_2x + b_2y &= c_2
\end{aligned}
$$

- The system has exactly <u>one solution</u> when $a_1b_2 - a_2b_1 \neq 0$
- The system has <u>no solution or infinitely many solutions</u> when $a_1b_2 - a_2b_1 = 0$

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The coefficient matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ has determinant $= a_1b_2 - a_2b_1$.

Remark. Determinant of the coefficient matrix determines the number of solutions of the given system. The system has a unique solution iff $D \neq 0$.

Application to linear equations

Solving the system by variable elimination:

$$\begin{aligned} a_1 b_2 x + b_1 b_2 y &= b_2 c_1 \\ a_2 b_1 x + b_1 b_2 y &= b_1 c_2 \end{aligned}$$

$$\begin{aligned} (a_1 b_2 - a_2 b_1)x &= b_2 c_1 - b_1 c_2 \\ x &= \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1} \end{aligned}$$

We have:

$$b_2 c_1 - b_1 c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = N_x \quad \text{and} \quad a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = D$$

Hence, $x = \frac{N_x}{D}$

Application to linear equations

Similarly, we can find the value of y:

$$a_1 a_2 x + a_2 b_1 y = a_2 c_1$$
$$a_1 a_2 x + a_1 b_2 y = a_1 c_2$$

$$(a_2 b_1 - a_1 b_2) y = a_2 c_1 - a_1 c_2$$
$$x = \frac{a_2 c_1 - a_1 c_2}{a_2 b_1 - a_1 b_2} = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

We have:

$$a_1 c_2 - a_2 c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = N_y \quad \text{and} \quad a_1 b_2 - a_2 b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = D$$

Hence, $y = \frac{N_y}{D}$

Example

Solve the following system using determinants:

$$\begin{cases} 3x - 4y = -10 \\ -x + 2y = 2 \end{cases}$$

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Solution:

$$N_x = \begin{vmatrix} -10 & -4 \\ 2 & 2 \end{vmatrix} = -20 - (-8) = -12$$

$$N_y = \begin{vmatrix} 3 & -10 \\ -1 & 2 \end{vmatrix} = 6 - 10 = -4$$

$$D = \begin{vmatrix} 3 & -4 \\ -1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence, $x = \frac{-12}{2} = -6$ and $y = \frac{-4}{2} = -2$.

Conclusion

Given:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

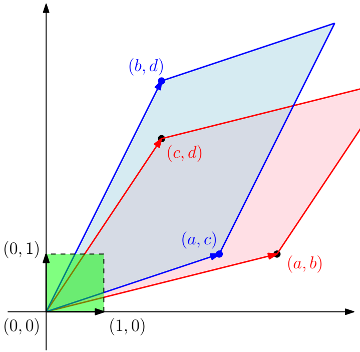
with the coefficient matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ having non-zero determinant (meaning that, the system has a unique solution).

The solution is given by:

$$x = \frac{N_x}{D} \quad \text{and} \quad y = \frac{N_y}{D}$$

where $N_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$, $N_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$, and $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Geometric interpretation



Matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be viewed as an "arrangement" of:

- row vectors: $\begin{bmatrix} a & b \end{bmatrix}$ and $\begin{bmatrix} c & d \end{bmatrix}$
- or, column vectors: $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$

The matrix defines the so-called *linear transformation* of the unit square (in green) formed by the *basis vectors* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, with respect to:

- the row vectors, shown by the red parallelogram; or
- the column vectors, shown by the blue parallelogram

Both parallelograms have the **same area**. Prove it!

Example

Given a matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$.

Draw two parallelograms that define the transformation of the unit square w.r.t. the row vectors and the column vectors, respectively.

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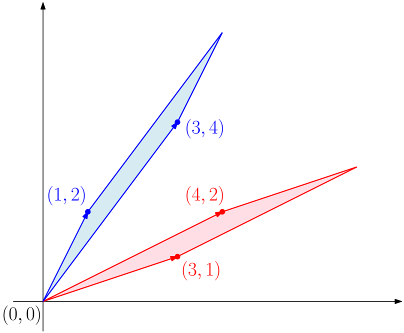
Solution:



Part 3: Determinants of 3×3 matrices

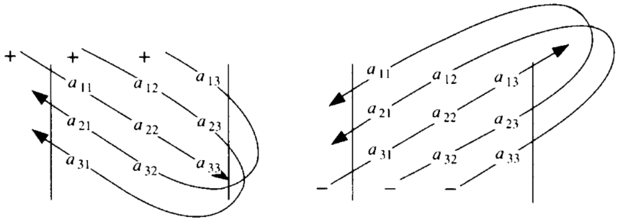
Determinants of matrices of order 3 (i.e., size 3×3)

Given a matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The determinant of the matrix above is defined as:

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$



Alternative form for the determinant of an order-3 matrix

The determinant of the matrix above is defined as:

$$
\begin{aligned}
\det(A) &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\
&= a_{11}(a_{22}a_{23} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}
\end{aligned}
$$

This formula can be illustrated as follows:

$$
a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}
$$

Example

Find the determinant of matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{bmatrix}$

Solution:

- *Using the diagram*

$$\det(A) = 3(5)(4) + 2(-1)(2) + (1)(-4)(-3) - 1(5)(2) - 2(-4)4 - 3(-1)(-3)$$
$$= 60 - 4 + 12 - 10 + 32 - 9 \ = \ 81$$

- *Using the alternative form*

$$\begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix} = 1 \begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 & 1 \\ -4 & 5 & -1 \\ 2 & -3 & 4 \end{vmatrix}$$
$$= 1 \begin{vmatrix} 5 & -1 \\ -3 & 4 \end{vmatrix} - 2 \begin{vmatrix} -4 & -1 \\ 2 & 4 \end{vmatrix} + 3 \begin{vmatrix} -4 & 5 \\ 2 & -3 \end{vmatrix}$$
$$= 1(20 - 3) - 2(-16 + 2) + 3(12 - 10) = 17 + 28 - 6 = 39$$

Applications to linear equations system

Given the following linear system:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{cases}$$

We can perform similar computations as in the case (2×2) matrix, in order to find a solution of the system.

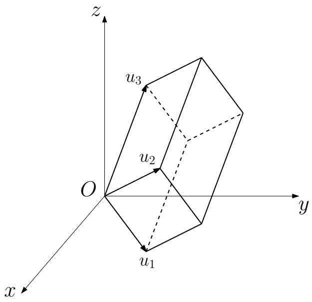
The coefficient matrix of the system is given by: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

The system has a unique solution only if $D = \det(A) \neq 0$.
The solution is given by:

$$x = \frac{N_x}{D}, \quad y = \frac{N_y}{D}, \quad z = \frac{N_z}{D}$$

where N_x, N_y, and N_z is obtained by replacing the 1st, 2nd, and 3rd column of A by the constant vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Geometric interpretation



In $\mathbb{R}^3$, the vectors u_1, u_2, and u_3 determine the parallelepiped,

which is the result of transforming the unit cube using the vectors $\{u_1, u_2, u_3\}$.

Remark.
Let $u_1, u_2, \ldots, u_n$ be vectors in $\mathbb{R}^n$. Then the parallelepiped is defined by:

$$S = \{a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \ : \ 0 \leq a_i \leq 1 \text{ for } i = 1, \ldots, n\}$$

with volume $V(S) =$ absolute value of $\det(A)$

Can you prove it?

Part 4: Determinants of arbitrary order (*a combinatorial way*)

Pattern in the determinant formulas

Can you find a pattern of the following determinant formulas?

- For 2×2 matrix: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

- For 3×3 matrix: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \end{aligned}$$

We will study these patterns!

Sign (parity) of a permutation

Given a sequence of elements: $\sigma = j_1 j_2 \ldots j_n$, a permutation of σ is defined as an arrangement of the objects in σ in a definite order.

The set of all permutations of n objects is denoted by S_n.

An inversion in σ is a pair of integers (i, k), such that $i > k$ but i precedes k in σ.

σ is called:

- even permutation, if there are an even number of inversions in σ;
- odd permutation, otherwise.

The sign or parity of the permutation σ is defined by:

$$\operatorname{sgn}(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is even} \\ -1 & \text{if } \sigma \text{ is odd} \end{cases}$$

Example: *sign of a permutation*

Given a permutation $\sigma = 35412$ in S_5. What is the sign of σ?

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Solution:

- 3 numbers (3, 4 and 5) precede 1;
- 3 numbers (3, 4 and 5) precede 2;
- 1 number (5) precedes 4;
- no number that precedes 3 or 4

Since $3 + 3 + 1 = 7$ is odd, then σ is an odd permutation. Hence

$$\text{sgn}(\sigma) = -1$$

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Exercises:

1. Find the sign of the permutation: $\epsilon = 123 \dots n$ in S_n.
2. Find the sign of each permutation in S_2 and S_3.
3. Is it true that in S_n, half of the permutations are even, and half of them are odd?

Using permutation in computing determinants (1)

Given an $n \times n$ matrix $A = [a_{ij}]$ over a field K.

Consider a product of n elements of A (here, $j_1 j_2 \ldots j_n$ is a permutation of $123 \ldots n$):

$$a_{1j_1} a_{2j_2} \ldots a_{nj_n}$$

such that:

- *one and only one* element comes from each row of A; and
- *one and only one* element comes from each column of A.

Q: How many different products of form $a_{1j_1} a_{2j_2} \ldots a_{nj_n}$ are there?

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Q: How many different products of form $a_{1j_1} a_{2j_2} \dots a_{nj_n}$ are there?

A: There are $n!$ such products, because there are $n!$ permutations of $j_1 j_2 \dots j_n$.

Using permutation in computing determinants (2)

The determinant of the $n \times n$ matrix $A = [a_{ij}]$ is defined as:

the sum of all the $n!$ products $a_{1j_1} a_{2j_2} \ldots a_{nj_n}$, where each product is multiplied by the sign of $\sigma = j_1 j_2 \ldots j_n$.

$$|A| = \sum_{\sigma} \text{sgn}(\sigma)\ a_{1j_1} a_{2j_2} \ldots a_{nj_n}$$

or, this can be written as:

$$|A| = \sum_{\sigma \in S_n} \text{sgn}(\sigma)\ a_{1\sigma(1)} a_{2j\sigma(2)} \ldots a_{n\sigma(n)}$$

Using permutation in computing determinants (3)

1. Given $A = [a_{11}]$, then $\det(A) = a_{11}$.
2. Given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$.
3. Given $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$$\begin{aligned}\det(A) = {} & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}\end{aligned}$$

to be continued...