Linear Algebra
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4.1 - Gaussian Elimination Algorithm

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Learning objectives

After this lecture, you should be able to:

- apply the Gaussian elimination algorithm to solve a system of linear equations.

Gaussian Elimination

System of linear equations

Given a system of m linear equations in n variables:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \quad (1)$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \quad (2)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad (3)$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \quad (4)$$

which can be represented as matrix:

$$\begin{bmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}x_1 & a_{m2}x_2 & \cdots & a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Linear system in matrix

or equivalently, written in matrix multiplication $\mathbf{Ax} = \mathbf{b}$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

This can be written shortly using augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Gaussian elimination algorithm

Gaussian elimination also known as row reduction.

Recall, three types of elementary row operations:

1. Swap the positions of two rows.
2. Multiply a row by a non-zero scalar.
3. Add to one row a scalar multiple of another.

Gaussian elimination algorithm (*cont.*)

Algorithm:

1. Represent the linear system with *augmented matrix*;
2. Perform the *elementary row operations* on the augmented matrix, so that row echelon matrix is obtained;

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & * \end{bmatrix}$$

3. Solve the echelon matrix using *backward substitution*.

Example of Gaussian elimination (*one solution*)

$$\begin{cases} 2x_1 + 3x_2 - x_3 = 5 \\ 4x_1 + 4x_2 - 3x_3 = 3 \\ -2x_1 + 3x_2 - x_3 = 1 \end{cases}$$

Solution:

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{R1/2} \begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R3+2R1]{R2-4R1} \begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & -2 & -1 & -7 \\ 0 & 6 & -2 & 6 \end{bmatrix} \xrightarrow{R2/(-2)} \begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 6 & -2 & 6 \end{bmatrix}$$

$$\xrightarrow{R3-6R2} \begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & -5 & -15 \end{bmatrix} \xrightarrow{R3/(-5)} \begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Example of Gaussian elimination (*cont.*)

From the augmented matrix, we obtain the following system:

$$\begin{cases} x_1 + \frac{3}{2}x_2 - \frac{1}{2}x_3 = \frac{5}{2} \\ x_2 + \frac{1}{2}x_3 = \frac{7}{2} \\ x_3 = 3 \end{cases}$$

Using the *backward substitution*, we obtain:

- From 3rd eq: $x_3 = 3$
- From 2nd eq:

$$x_2 + \frac{1}{2}x_3 = \frac{7}{2} \;\rightarrow\; x_2 = \frac{7}{2} - \frac{1}{2}(3) = 2$$

- From 1st eq:

$$x_1 + \frac{3}{2}x_2 - \frac{1}{2}x_3 = \frac{5}{2} \;\rightarrow\; x_1 = \frac{5}{2} - \frac{3}{2}(2) - \frac{1}{2}(3) = 1$$

The solution is: $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

Example 2 (*parametric solution*)

$$\begin{cases} x_1 - x_2 + 2x_3 = 5 \\ 2x_1 - 2x_2 + 4x_3 = 10 \\ 3x_1 - x_2 + 6x_3 = 15 \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 2 & -2 & 4 & 10 \\ 3 & -1 & 6 & 15 \end{bmatrix} \xrightarrow[R3-3R1]{R2-2R1} \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the augmented matrix, we can only derive one equation:

$$x_1 - x_2 + 2x_3 = 5 \ \rightarrow \ x_1 = 5 + x_2 - 2x_3$$

Assign: $x_2 = r$ and $x_3 = s$, where $r, s \in \mathbb{R}$.

The solution: $x_1 = 5 + r - 2s, \ \ x_2 = r, \ \ x_3 = s, \ \ \text{with } r, s \in \mathbb{R}$.

Example 3 (*parametric solution*)

Given the following system:

$$\begin{cases} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ 2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1 \\ 5x_3 + 10x_4 + 15x_6 = 5 \\ 2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6 \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 3 (*cont.*)

From the last augmented matrix, we obtain:

$$\begin{cases} x_1 + 3x_2 - 2x_3 \qquad + 2x_5 \qquad = 0 \\ \qquad\qquad x_3 + 2x_4 \qquad + 3x_6 = 1 \\ \qquad\qquad\qquad\qquad\qquad x_6 = 1/3 \end{cases}$$

- From the 3rd equation: $x_6 = 1/3$
- Substitute to the 2nd equation: $x_3 + 2x_4 + 3x_6 = 1$

$$\begin{aligned} \Rightarrow\ x_3 &= 1 - 2x_4 - 3x_6 = 1 - 2x_4 - 3(1/3) \\ &= 1 - 2x_4 - 1 = -2x_4 \end{aligned}$$

- Substitute to the 1st equation: $x_1 + 3x_2 - 2x_3 + 2x_5 = 0$

$$\begin{aligned} \Rightarrow\ x_1 &= -3x_1 + 2x_3 - 2x_5 = -3x_2 + 2(-2x_4) - 2x_5 \\ &= -3x_2 - 4x_4 - 2x_5 \end{aligned}$$

Let $x_2 = r$, $x_4 = s$, $x_5 = t$, where $r, s, t \in \mathbb{R}$. Then:

$$x_1 = -3r - 4s - 2t,\ x_2 = r,\ x_3 = -2s,\ x_4 = s,\ x_5 = t,\ x_6 = 1/3$$

Example 4 (*no solution*)

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 = 2 \\ 3x_1 + 4x_2 + x_3 = 2 \end{cases}$$

Solution:

$$\begin{bmatrix} 0 & -2 & 3 & 1 \\ 3 & 6 & -3 & -2 \\ 6 & 6 & 3 & 5 \end{bmatrix} \xrightarrow[R3-3R1]{R2-2R1} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & -2 & -1 \end{bmatrix}$$

$$\xrightarrow{R2/(-2)} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -2 & -1 \end{bmatrix} \xrightarrow{R3+2R2} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

From the last augmented matrix, we obtain the following system:

$$\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ x_2 + x_3 = 0 \\ 0x_1 + 0x_2 + 0x_3 = -1 \end{cases}$$

From the 3rd equation, no value for x_1, x_2, and x_3 can satisfy the equation. So, the system has no solution

Exercise

Solve the following system using the Gaussian elimination method:

$$\begin{cases} \quad\quad - 2x_2 + 3x_3 = 1 \\ 3x_1 + 6x_2 - 3x_3 = -2 \\ 6x_1 + 6x_2 + 3x_3 = 5 \end{cases}$$