Linear Algebra

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# 4.2 - Gauss-Jordan Elimination

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# Learning objectives

After this lecture, you should be able to:

- apply the Gauss-Jordan elimination algorithm to solve a system of linear equations.

 

# Introduction

Carl Friedrich Gauss
(*German mathematician*)

Wilhelm Jordan
(*German mathematician*)

# Introduction

This is a development of the Gaussian-Elimination method.

- The ERO is implemented on the augmented matrix, so that a *reduced echelon matrix* is obtained.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & * & * & \cdots & * & * \\ 0 & 1 & * & \cdots & * & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & * \end{bmatrix}$$

- The difference with the Gaussian method is that, here backward substitution is not needed to obtain the variables values.
- The value of each variable can be derived directly from the augmented matrix.

 

# Steps in Gauss-Jordan method

1. **Forward phase (Gauss elimination phase)**

   Under the main diagonal of 1's should be 0.

$$\begin{bmatrix} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & -1 & 1 \end{bmatrix} \xrightarrow{ERO} \begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

2. **Backward phase**

   Above the main diagonal of 1's should be 0.

$$\begin{bmatrix} 1 & 3/2 & -1/2 & 5/2 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \overset{R1-(3/2)R2}{\sim} \begin{bmatrix} 1 & 0 & -5/4 & -11/4 \\ 0 & 1 & 1/2 & 7/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\overset{\substack{R1+(5/4)R3 \\ R2-(1/2)R3}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The last matrix is a reduced row echelon form.

We can derive the solution directly: $x_1 = 1, \; x_2 = 2, \; x_3 = 3.$

# Example 1

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 = -1 \\ 2x_1 + x_2 - 2x_3 - 2x_4 = -2 \\ -x_1 + 2x_2 - 4x_3 + x_4 = 1 \\ 3x_1 - 3x_4 = -3 \end{cases}$$

**Solution:**

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix} \underset{\sim}{\begin{matrix} R2-2R1 \\ R4-3R1 \\ R3+R1 \end{matrix}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R2/3} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{bmatrix} \underset{\sim}{\begin{matrix} R4-3R2 \\ R3-R2 \end{matrix}} \begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R1+R2} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding equations are:

$$x_1 - x_4 = -1$$

$$x_2 - 2x_3 = 0$$

# Example 1 (*cont.*)

The last augmented matrix is in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution can be obtained by solving the system:

$$x_1 - x_4 = -1$$
$$x_2 - 2x_3 = 0$$

From the 2nd eq, we obtain: $x_2 = 2x_3$
From the 1st eq, we obtain: $x_1 = x_4 - 1$

Let $x_3 = r$ and $x_4 = s$ with $r, s \in \mathbb{R}$.
Then the solution of the system is:

$$x_1 = s - 1,\ x_2 = 2r,\ x_3 = r,\ x_4 = s$$

# Example 2

Solve the following system using Gauss-Jordan method

$$\begin{cases} \quad\quad\quad\quad\; - \;\; 2x_3 \quad\quad\;\; + \;\; 7x_5 = 12 \\ 2x_1 + 4x_2 - 10x_3 + 6x_4 + 12x_5 = 28 \\ 2x_1 + 4x_2 - \;\; 5x_3 + 8x_4 - \;\; 5x_5 = -1 \end{cases}$$

**Solution:**

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \overset{R1 \leftrightarrow R2}{\sim} \begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \overset{R1/2}{\sim}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \overset{R3 - 2R1}{\sim} \begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \overset{R2/(-2)}{\sim}$$

# Example 2 (*cont.*)

$$\begin{bmatrix}1 & 2 & -5 & 3 & 6 & 14\\0 & 0 & 1 & 0 & -7/2 & -6\\0 & 0 & 5 & 0 & -17 & -29\end{bmatrix} \overset{R3-5R2}{\sim} \begin{bmatrix}1 & 2 & -5 & 3 & 6 & 14\\0 & 0 & 1 & 0 & -7/2 & -6\\0 & 0 & 0 & 0 & 1/2 & 1\end{bmatrix} \overset{R3/(1/2)}{\sim}$$

$$\begin{bmatrix}1 & 2 & -5 & 3 & 6 & 14\\0 & 0 & 1 & 0 & -7/2 & -6\\0 & 0 & 0 & 0 & 1 & 2\end{bmatrix} \underset{R2+7/2R3}{\overset{R1-6R3}{\sim}} \begin{bmatrix}1 & 2 & -5 & 3 & 0 & 2\\0 & 0 & 1 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 1 & 2\end{bmatrix} \overset{R1+5R2}{\sim}$$

$$\begin{bmatrix}1 & 2 & 0 & 3 & 0 & 7\\0 & 0 & 1 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 1 & 2\end{bmatrix}$$

From the last augmented matrix, can be derived:

$$\begin{cases} x_1 + 2x_2 + 3x_4 = 7 \\ x_3 = 1 \\ x_5 = 2 \end{cases}$$

Let $x_2 = s$ and $x_4 = t$, the solution of the system:

$x_1 = 7 - 2s - 3t,\ x_2 = s,\ x_3 = 1,\ x_4 = t,\ x_5 = 2,\ s, t \in \mathbb{R}$

 

# Example 3

Solve the following system using Gauss-Jordan method

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$$

The augmented matrix:

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

which can be reduced into row-echelon form:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Example 3 (*cont.*)

The corresponding linear equations system is:

$$\begin{cases} x_1 + x_2 \qquad\quad + x_5 = 0 \\ \qquad\qquad x_3 \quad + x_5 = 0 \\ \qquad\qquad\quad x_4 \qquad = 0 \end{cases}$$

Solving the *leading variables*, we get:

$$x_1 = -x_2 - x_5$$
$$x_3 = -x_5$$
$$x_4 = 0$$

The general solution is:

$$x_1 = -s - t,\ x_2 = s,\ x_3 = -t,\ x_4 = 0,\ x_5 = t, \quad \text{with } s, t \in \mathbb{R}$$

 

# Analysis of Example 3

What can you observe from the above linear system?

$$\begin{cases} 2x_1 + 2x_2 - x_3 + x_5 = 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 = 0 \\ x_1 + x_2 - 2x_3 - x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{cases}$$

 

# Homogeneous Linear System

# Homogeneous Linear System

Recall that the following system is called homogeneous linear system.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots&\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

The solution always has a solution, namely:

$$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0$$

this solution is called trivial solution.

If a solution other than $x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0$ exists, then it is called non-trivial solution.

 

# Homogeneous Linear System

Example

From Example 3 of the previous section, we obtain the solution of the given linear system is:

$$x_1 = -s - t,\ x_2 = s,\ x_3 = -t,\ x_4 = 0,\ x_5 = t, \quad \text{with } s, t \in \mathbb{R}$$

Here, if $s, t = 0$, then we get the **trivial solution**, namely:

$$x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 0,\ x_5 = 0$$

We can set $s \neq 0$ or $t \neq 0$ to get **non-trivial solutions**.

# Example

Solve the following homogeneous linear system by Gauss-Jordan elimination:

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix}$$

**Solution:**

$$\begin{bmatrix} 0 & 2 & 2 & 4 & 0 \\ 1 & 0 & -1 & -3 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \overset{R1 \leftrightarrow R2}{\sim} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 2 & 3 & 1 & 1 & 0 \\ -2 & 1 & 3 & -2 & 0 \end{bmatrix} \underset{R4+2R1}{\overset{R3-2R1}{\sim}} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 2 & 2 & 4 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix}$$

$$\overset{R2/2}{\sim} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 3 & 3 & 7 & 0 \\ 0 & 1 & 1 & -8 & 0 \end{bmatrix} \underset{R4-R2}{\overset{R3-3R2}{\sim}} \begin{bmatrix} 1 & 0 & -1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix} \underset{\substack{R2-2R3 \\ R4+10R3}}{\overset{R1+3R3}{\sim}} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Analysis

Why the following matrix has a reduced-row echelon form?

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

# Analysis of homogeneous system

*When does a homogeneous linear system has an infinitely many solutions?*

 

# Analysis of homogeneous system

*When does a homogeneous linear system has an infinitely many solutions?*

*How many free variables do exist in a homogeneous linear system (relate it to the value of $n$ and $r$)?*

# Analysis of homogeneous system

**Theorem**

*If a homogeneous linear system has $n$ unknowns, and if the reduced row echelon form of its augmented matrix has $r$ nonzero rows, then the system has $n - r$ free variables.*

**Theorem**

*A homogeneous linear system with more unknowns than equations has infinitely many solutions.*