Linear Algebra

[KOMS120301] - 2023/2024

2.1 - Algebra of Matrices

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Motivating example (1)

$$
\begin{array}{c c}
 & \begin{array}{ccccc} \text{Mon} & \text{Tue} & \text{Wed} & \text{Thu} & \text{Fri} \end{array} \\
\begin{array}{c} \text{John} \\ \text{Amy} \\ \text{Bob} \end{array} & \begin{pmatrix} 30 & 10 & 20 & 9 & 14 \\ 10 & 9 & 7 & 19 & 25 \\ 20 & 7 & 0 & 10 & 20 \end{pmatrix}
\end{array}
$$

A matrix of messages

Motivating example (2)

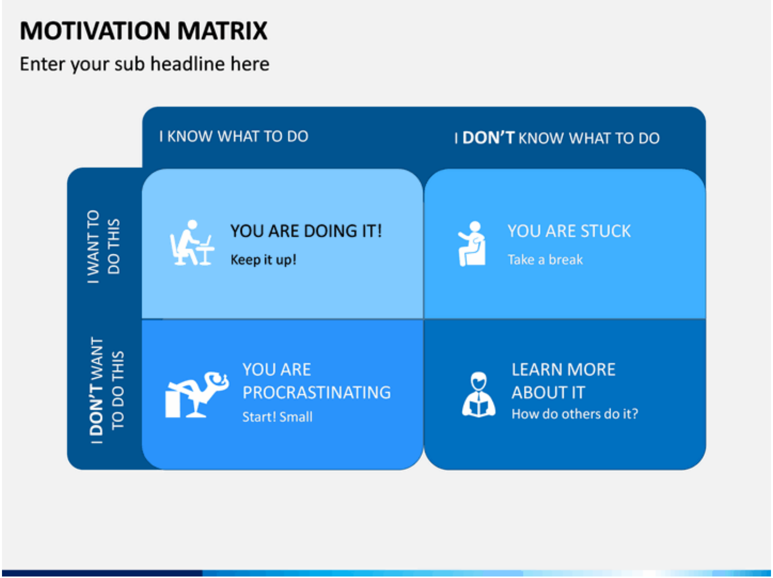
$$
\begin{array}{c c}
 & \begin{array}{ccccc} \text{Jan} & \text{Feb} & \text{Mar} & \text{Apr} & \text{May} \end{array} \\
\begin{array}{c} \text{Rent} \\ \text{Grocery} \\ \text{Car} \end{array} &
\begin{pmatrix}
1000 & 1000 & 1050 & 1050 & 1050 \\
300 & 250 & 350 & 310 & 305 \\
400 & 450 & 350 & 300 & 320
\end{pmatrix}
\end{array}
$$

A matrix of expenses

Motivating example (3)

$$
\begin{array}{c c}
 & \begin{array}{ccc} \text{Boston} & \text{New York} & \text{London} \end{array} \\
\begin{array}{c} \text{Boston} \\ \text{New York} \\ \text{London} \end{array} & \begin{pmatrix} 0 & 187 & 3269 \\ 187 & 0 & 3459 \\ 3269 & 3459 & 0 \end{pmatrix}
\end{array}
$$

Motivating example (4)



Then...what can you say about matrix?

Learning objectives

After this lecture, you should be able to:

1. Define and write the components of a matrix (row, column, diagonal, and entry) correctly.
2. Perform the operations between matrices, such as: scalar multiplication, matrix addition, matrix mutiplication, transpose, powering of matrix, and polynomial of matrix.
3. Apply the properties of matrix operations to solve a problem.
4. Explain the concept and properties of square matrix.
5. Apply the concept of block matrices to solve matrix operation.

Part 1: Matrices and their operations

Formal definition of matrices

A matrix A over a field K (or simply a matrix A, when K is implicit), is a rectangular array of scalars:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The rows of matrix A are the m horizontal lists:

$$(a_{11}, a_{12}, \ldots, a_{1n}),\ (a_{21}, a_{22}, \ldots, a_{2n}),\ \ldots,\ (a_{m1}, a_{m2}, \ldots, a_{mn})$$

The columns of matrix A are the n vertical lists:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m3} \end{bmatrix}, \ldots, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}$$

Note: So, a matrix is composed by a set of *vectors*.

Formal definition of matrices

The element a_{ij} of matrix A (on row i, column j) is called ij-entry or ij-element.

We write the matrix as: $A = [a_{ij}]$.

A is a matrix of size $m \times n$.

- if $m = 1$ (only one row), then it is called row matrix or row vector;
- if $n = 1$ (only one column), then it is called column matrix or column vector.

A is called zero matrix if all entries of the matrix are zero.

Example

- Row matrix: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$
- Column matrix: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$
- Zero matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
- A 3×2 matrix: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Matrix operations

We are going to discuss:

1. Scalar multiplication
2. Matrix addition
3. Matrix multiplication
4. Transpose matrix
5. Power of matrix
6. Polynomial of matrix

1. Scalar multiplication

The product of matrix $A = [a_{ij}]$ with a scalar $k \in \mathbb{R}$ is defined as:

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

Moreover, $-A = (-1)A$.

2. Matrix addition

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be matrices of the same size $m \times n$. The sum of A and B is defined as:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

Moreover, $A - B = A + (-B)$.

Properties of matrices under addition and scalar multiplication

Theorem

Let A, B, and C be matrices with the same size, and $k, k' \in \mathbb{R}$. Then:

- $(A + B) + C = A + (B + C)$ *(associativity)*
- $A + B = B + A$ *(commutativity)*
- $A + 0 = A$ *(0 is the identity elt over addition)*
- $A + (-A) = 0$ *(invers matrix over addition)*
- $k(A + B) = kA + kB$ *(distributivity)*
- $(k + k')A = kA + k'A$ *(distributivity w.r.t. scalar)*
- $(kk')A = k(k'A)$ *(associativity w.r.t. scalar)*
- $1 \cdot A = A$ *(1 is the identity elt over scalar multiplication)*

Note: Hence, the sum $A_1 + A_2 + \cdots + A_n$ can be done in any order, and does not require any parenthesis.

Example

Given the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 5 & 5 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 2 \\ -1 & 0 & 4 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 9 & 8 & 7 \end{bmatrix}$$

Simplify the following matrix expression.

- $A + B$
- $B - C$
- $-3A + 2B$
- $5A + 2B - 3C$
- $3(A - C) + B$
- $A - A$

3. Matrix multiplication

Special case: the product of a row matrix and a column matrix having the same number of elements.

Let $A = [a_i]$ be a row matrix and $B = [b_i]$ be a column matrix. Then the product AB is defined as:

$$AB = [a_1 \;\; a_2 \;\; \ldots \;\; a_n] \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

Example

$$[7, -4, 5] \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 7(3) + (-4)(2) + 5(-1) = 21 - 8 - 5 = 8$$

(or this can be written as [8])

Matrix multiplication

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of size $m \times p$ and $p \times n$ respectively. Then the product of A and B is a matrix AB of size $m \times n$ defined by:

$$\begin{bmatrix} a_{11} & \cdots & a_{1p} \\ \cdot & \cdots & \cdot \\ a_{i1} & \cdots & a_{ip} \\ \cdot & \cdots & \cdot \\ a_{m1} & \cdots & a_{mp} \end{bmatrix} \times \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot & \cdots & \cdot \\ b_{p1} & \cdots & b_{pj} & \cdots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \cdots & \cdot \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}$$

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$

Example

Find AB where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$.

Example

Find AB where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$.

Multiply each row of A with each column of B.

Since A is of size 2×2 and B is of size 2×3, then AB is of size 2×3.

$$AB = \begin{bmatrix} 2+15 & 0-6 & -4+18 \\ 4-5 & 0+2 & -8-6 \end{bmatrix} = \begin{bmatrix} 17 & -6 & 14 \\ -1 & 2 & -14 \end{bmatrix}$$

Relation between matrix addition and matrix multiplication

Theorem

Let A, B, and C be matrices. Then whenever the products and sums are defined,

- $(AB)C = A(BC)$ *(associative)*
- $A(B + C) = AB + AC$ *(left distributive)*
- $(B + C)A = BA + CA$ *(right distributive)*
- $k(AB) = (kA)B = A(kB)$ *where* $k \in \mathbb{R}$
- $0A = 0$ *and* $A0 = 0$, *where* 0 *is the zero matrix*

Transpose matrix

The transpose of a matrix A, denoted by A^T, is the the matrix obtained by writing the columns of A, in order, as rows.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \text{ then } A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Note: If A has size $m \times n$, then A^T has size $n \times m$.

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -3 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}$$

Operations on matrix transpose

Theorem

If A and B are matrices such that the following operations are well defined, then:

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(A - B)^T = A^T - B^T$
4. $(kA)^T = kA^T$
5. $(AB)^T = B^T A^T$

Note: If A has order $m \times n$, then A^T has order $n \times m$.

Powers of Matrices, Polynomials in Matrices

Let A be an n-square matrix over $\mathbb{R}$ (or over other fields). Powers of A are defined as:

$$A^2 = AA, \quad A^3 = A^2A, \quad \dots, \quad A^{n+1} = A^nA, \quad \dots, \quad \text{and} \quad A^0 = 1$$

We can also define polynomials in the matrix A. For any polynomial:

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \quad \text{where } a_i \in \mathbb{R},$$

Polynomial $f(A)$ is defined as:

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$$

Note: If $f(A) = 0$ (the zero matrix), then A is called a *zero* or *root* of $f(x)$.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$. Then:

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix}, \quad \text{and}$$

$$A^3 = A^2 A = \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -11 & 38 \\ 57 & -106 \end{bmatrix}$$

Suppose $f(x) = 2x^2 - 3x + 5$, then:

$$f(A) = 2 \begin{bmatrix} 7 & -6 \\ -9 & 22 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & -18 \\ -27 & 61 \end{bmatrix}$$

Exercise

1. Form a group of 3 students;
2. Do the following exercises (Howard Anton's book):
   - Number 1 & 2 (2 questions ©)
   - Number 3-6 (3 questions ©)
   - Number 7-8

Part 2: Square matrices

Square matrices

A square matrix is a matrix with the same number of rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Diagonal and Trace

Let $A = [a_{ij}]$ be an n-square matrix. The diagonal or main diagonal of A consists of the elements with the same subscripts, that is:

$$a_{11},\ a_{22},\ \ldots,\ a_{nn}$$

The trace of A, denoted by $\mathrm{tr}(A)$ is the sum of the diagonal elements of A.

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

Theorem (Properties of trace)

- $tr(A + B) = tr(A) + tr(B)$
- $tr(kA) = k\ tr(A)$
- $tr(A^T) = tr(A)$
- $tr(AB) = tr(BA)$ *(recall that $AB \neq BA$ is not always correct)*

Identity matrix, scalar matrices

The identity or unit matrix, denoted by I_n (or simply I) is the square matrix $n \times n$, with 1's on the diagonal, and 0's elsewhere.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

I has a similar role as the scalar 1 for $\mathbb{R}$.

Important property: When it is well-defined,

$$IA = A$$

For some scalar $k \in \mathbb{R}$, the matrix kI is called scalar matrix corresponding to scalar k.

Special types of square matrices

A matrix $D = [d_{ij}]$ is a diagonal matrix if its nondiagonal entries are all zero.

$$D = \text{diag}(d_{11}, d_{22}, \ldots, d_{nn})$$

where some or all the d_{ii} may be zero.

Example

$$\begin{bmatrix} 3 & 0 & \cdots & 0 \\ 0 & -5 & \cdots & 0 \\ \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & \cdots & 9 \end{bmatrix}$$

Hence, identity matrices and scalar matrices are also diagonal matrices.

Upper and lower triangular matrices

A *square matrix* $A = [a_{ij}]$ is upper triangular, if all entries *below* the (main) diagonal are equal to 0.

A lower triangular matrix is a square matrix whose entries *above* the diagonal are all zero.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \dots & \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Upper triangular matrix (*left)* and lower triangular matrix (*right*)

Property of upper and lower triangular matrices

Theorem

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are $n \times n$ triangular matrices. Then:

$$A + B, \quad kA, \quad AB$$

are triangular matrices w.r.t. diagonals:

$$(a_{11}+b_{11}, \ldots, a_{nn}+b_{nn}), \ (ka_{11}, \ldots, ka_{nn}), \ (a_{11}b_{11}, \ldots, a_{nn}b_{nn})$$

Symmetric matrices

A matrix A is symmetric if $A^T = A$, i.e. $a_{ij} = a_{ji}$ for every $i, j \in \{1, 2, \ldots, n\}$.

It is skew-symmetric if $A^T = -A$.

Example

$$A = \begin{bmatrix} 2 & -3 & 5 \\ -3 & 6 & 7 \\ 5 & 7 & -8 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 3 & -4 \\ -3 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$$

A is a symmetric matrix, and B is a skew-symmetric matrix.

Can you find other examples? Find an example of matrix that is neither symmetric nor skew-symmetric.

Normal matrices

A matrix A is normal if $AA^T = A^T A$.

Example

Let $A = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix}$. Then:

$$AA^T = \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 6 & 3 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 45 & 0 \\ 0 & 45 \end{bmatrix}$$

Since $AA^T = A^T A$, the matrix A is normal.

Exercise of square matrices

- Create 3 groups;
- Each group discusses about the application of the following matrices in CS:
  - Upper (lower) triangular matrices;
  - Symmetric matrices;
  - Normal matrices.
- Write your discussion's result on a piece of paper (i.e., 1 page), and submit it.

Part 3: Block matrices

Block matrices

Using a system of horizontal and vertical (dashed) lines, a matrix A can be partitioned into submatrices called blocks (or cells) of A.

Example

$$\left(\begin{array}{cc|cc|c} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right) \quad \left(\begin{array}{cc|ccc} 1 & -2 & 0 & 1 & 3 \\ \hline 2 & 3 & 5 & 7 & -2 \\ 3 & 1 & 4 & 5 & 9 \\ \hline 4 & 6 & -3 & 1 & 8 \end{array}\right) \quad \left(\begin{array}{ccc|cc} 1 & -2 & 0 & 1 & 3 \\ 2 & 3 & 5 & 7 & -2 \\ \hline 3 & 1 & 4 & 5 & 9 \\ 4 & 6 & -3 & 1 & 8 \end{array}\right)$$

Operations on block matrices

Let $A = [A_{ij}]$ and $B = [B_{ij}]$ are block matrices with the same numbers of row and column blocks, and suppose that corresponding blocks have the same size.

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1n} + B_{1n} \\ A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1n} + B_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & \cdots & A_{mn} + B_{mn} \end{bmatrix}$$

and

$$kA = \begin{bmatrix} kA_{11} & kA_{12} & \cdots & kA_{1n} \\ kA_{21} & kA_{22} & \cdots & kA_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ kA_{m1} & kA_{m2} & \cdots & kA_{mn} \end{bmatrix}$$

Square block matrices

A block matrix M is called a square block matrix if:

1. M is a square matrix.
2. The blocks (seen as entries) form a square matrix.
3. The diagonal blocks are also square matrices.

Example

$$A = \left(\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 9 & 8 & 7 & 6 & 5 \\ \hline 4 & 4 & 4 & 4 & 4 \\ 3 & 5 & 3 & 5 & 3 \end{array}\right) \qquad B = \left(\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 9 & 8 & 7 & 6 & 5 \\ 4 & 4 & 4 & 4 & 4 \\ \hline 3 & 5 & 3 & 5 & 3 \end{array}\right)$$

Which one of the matrices is a square block matrix?

Square block matrices

A block matrix M is called a square block matrix if:

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2. The blocks (seen as entries) form a square matrix.
3. The diagonal blocks are also square matrices.

Example

$$A = \left(\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 9 & 8 & 7 & 6 & 5 \\ \hline 4 & 4 & 4 & 4 & 4 \\ 3 & 5 & 3 & 5 & 3 \end{array}\right) \qquad B = \left(\begin{array}{cc|cc|c} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ \hline 9 & 8 & 7 & 6 & 5 \\ 4 & 4 & 4 & 4 & 4 \\ \hline 3 & 5 & 3 & 5 & 3 \end{array}\right)$$

Which one of the matrices is a square block matrix?

B is a square block matrix.

Block diagonal matrices

A block diagonal matrix is a square block matrix $M = [A_{ij}]$ s.t. the non-diagonal blocks are zero matrices.

Example

$$\left(\begin{array}{cc|cc|c} 1 & 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 7 & 6 & 0 \\ 0 & 0 & 4 & 4 & 0 \\ \hline 0 & 0 & 0 & 0 & 3 \end{array}\right)$$

A block diagonal matrix is often denoted as $M = \mathrm{diag}(A_{11}, A_{22}, \ldots, A_{rr})$

Determinants and inverses of small matrices

The square matrix A is said to be invertible or non-singular if $\exists B$ s.t.:

$$AB = BA = I \qquad \text{where } I \text{ is the identity matrix}$$

Note: The matrix B is single (exactly one inverse), and is called the inverse of A, which is denoted by A^{-1}. The relationship A and B is **symmetric**:

If B is the inverse of A, then A is the inverse of B, i.e.

$$(A^{-1})^{-1} = A$$

Example

Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ dan $B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ Hence:

$$AB = \begin{bmatrix} 6-5 & -10+10 \\ 3-3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So A and B are inverses.

Practice and review

Given the following matrix:

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad \text{dan} \quad \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

- Is B the inverse of A?
- Is A the inverse of B?

Solution.

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{dan} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So A and B are inverses.

Question. Can you find two square matrices A and B of size 2×2, where B is the inverse of A but A is not the inverse of B?

Practice and review

Think back to your lessons in high school.

Find the inverse of:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \quad \text{dan} \quad B = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

Solution.

$|A| = 2 \cdot 5 - 3 \cdot 4 = 10 - 12 = -2$. Since $|A| \neq 0$, then matrix A has an inverse.

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{3}{2} \\ 2 & -1 \end{bmatrix}$$

Meanwhile, $|B| = 1 \cdot 6 - 3 \cdot 2 = 6 - 6 = 0$. So the matrix B does not have an inverse or is a singular matrix.

Exercise

(Practice this at home!)

1. Find an algorithm for matrix multiplication

Given two matrices:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -2 \\ 3 & 1 & 9 \\ 4 & 6 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 1 & 4 \\ -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 4 \end{pmatrix}$$

- Compute $A \times B$.
- Describe the step-by-step procedure to compute $A \times B$ for any matrix $A_{m \times k}$ and $B_{k \times n}$.
- Write the procedure in algorithm (you may write it as a pseudocode).

2. How to solve matrix multiplication using block matrix?

Given two matrices:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 3 & -2 \\ 3 & 1 & 9 \\ 4 & 6 & 8 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 2 & 1 & 4 \\ -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 4 \end{pmatrix}$$

Compute $A \times B$.

What if the two matrices are written in block matrices?

$$A = \left(\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & 3 & -2 \\ \hline 3 & 1 & 9 \\ 4 & 6 & 8 \end{array}\right) \qquad B = \left(\begin{array}{cc|cc} 0 & 2 & 1 & 4 \\ -1 & 1 & 0 & 0 \\ \hline 2 & 3 & -1 & 4 \end{array}\right)$$

Can you derive the step-by-step of block matrix multiplication?

to be continued...