Linear Algebra

[KOMS119602] - 2022/2023

10 - Subspace

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Learning objectives

After this lecture, you should be able to:

1. explain the concept of subspace;
2. analyze if a given set of vectors in a vector space is a subspace of the vector space.

Subspace

Subspace

Let V be a vector space. A set $W \subseteq V$ is a subspace of V, if W is a vector space w.r.t. the addition and scalar multiplication operations defined on V.

Example: Let $V = \mathbb{R}^3$ and W is a plane that go through the point $(0, 0, 0)$.

Proof.

W should have a function: $ax + by + cz = 0$.

- *Closure:* Let $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$ be points in W, and $k \in \mathbb{R}$. Then:
  - $\mathbf{u} + \mathbf{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2) \in W$, because $(x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 0$.
  - $k\mathbf{u} = (kx_1, ky_1, kz_1) \in W$ because $kx_1 + kx_2 + kx_3 = 0$.
- *Identity:* The zero element is $\mathbf{0} = (0, 0, 0)$ and the one element is 1. Clearly, $\mathbf{0} + \mathbf{u} = \mathbf{u}$ and $1\mathbf{u} = \mathbf{u}$, for every $\mathbf{u} \in W$.
- The *inverse* of $\mathbf{u} = (x_1, y_1, z_1)$ is $-\mathbf{u} = (-x_1, -y_1, -z_1)$. Clearly, $\mathbf{u} = (-\mathbf{u}) = \mathbf{0}$.
- Clearly, the commutative, associative, and distributive properties are satisfied.

Subspace theorem

Theorem

Let V be a vector space. If W is a set containing at least one vector of V, then W is a subspace of V iff the following conditions hold.

1. *If* $\mathbf{u}, \mathbf{v} \in W$, *then* $(\mathbf{u} + \mathbf{v}) \in W$.
2. *If* k *is a scalar, and* $\mathbf{u} \in W$, *then* $k\mathbf{u} \in W$.

By this theorem, then to check that W is a subspace of V, it is enough to check only **Axiom 1** (closed under addition and closed under scalar multiplication properties).

Subspace theorem (*cont.*)

Proof.

Since V is a vector space, then the axioms: *commutativity, associativity, identity, inverse,* and *distributivity* are satisfied.

Since the properties hold for every vector in V, then they hold for the subset W.

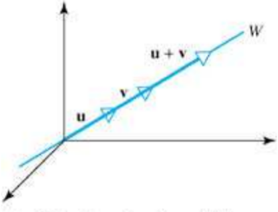
It is enough to check the *closure* property.

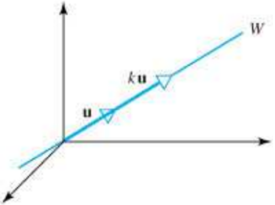
$\square$

Example of subspace (1)

A line through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, with vector addition and scalar multiplication operations, is a subspace of $\mathbb{R}^3$.

Geometric proof





Let L be a line goes through the origin of $\mathbb{R}^3$. Given two vectors $\mathbf{u}, \mathbf{v} \in L$. Clearly, the vectors:

$$(\mathbf{u} + \mathbf{v}) \quad \text{and} \quad k\mathbf{u}, \; k \in \mathbb{R}$$

lie on the line (they are vectors with the same direction, but different magnitudes). So the closure property is satisfied.

Example of subspace (2) (*cont.*)

Exercise: *Algebraic proof*

Algebraically, prove that a line through the origin of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$, with vector addition and scalar multiplication operations, is a subspace of $\mathbb{R}^3$.

Example of subspace (2)

The set of points on the plane that goes through the origin in $\mathbb{R}^3$, with vector addition and scalar multiplication operations, is a subspace of $\mathbb{R}^3$.

The set of points that go through the origin of $\mathbb{R}^3$ has function:

$$ax + by + cz = 0$$

Check if the addition and scalar multiplication properties are satisfied.

1. Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be vectors in $\mathbb{R}^3$. Then:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

   Clearly,

$$\begin{aligned} a(u_1 + v_1) &+ b(u_2 + v_2) + c(u_3 + v_3) \\ &= (au_1 + bu_2 + cu_3) + (av_1 + bv_2 + cv_3) = 0 + 0 = 0 \end{aligned}$$

Example of non-subspace

The set W of all points (x, y) in $\mathbb{R}^2$ s.t. $x \geq 0$ and $y \geq 0$, cannot be a subspace of $\mathbb{R}^3$.

W is not closed under scalar multiplication. For example:

$$\mathbf{v} = (1, 1) \in W \quad \text{but} \quad (-1)\mathbf{v} = -\mathbf{v} = (-1, -1) \notin W$$

Please read the materials and do the relevant exercises in the Howard Anton's book