Width Parameters on Even-Hole-Free Graphs

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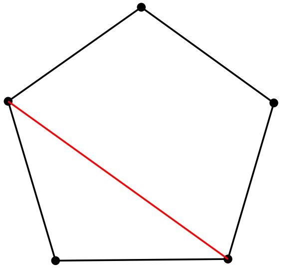
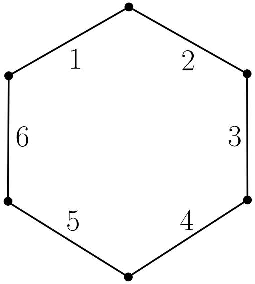
Ch 1: Introduction

Even hole





cycle even hole

Even-hole-free (EHF): the graph does not contain even hole, as an induced subgraph.

Induced subgraph, **even-hole-free graphs**

- H is an induced subgraph of G if H can be obtained from G by **deleting vertices** (denoted by $H \subseteq_{\text{ind}} G$)

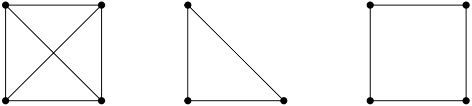


Figure: G, $H \subseteq_{\text{ind}} G$, $H \not\subseteq_{\text{ind}} G$

- G is H-free if no induced subgraph of G is isomorphic to H
- When $\mathcal{F}$ is a family of graphs, $\mathcal{F}$-free means H-free, $\forall H \in \mathcal{F}$

Tree-width

Tree-width ($tw(G)$: *a parameter measuring how similar a graph from being a tree*

$$\mathrm{tw}(G) = \min_{H \text{ chordalization of } G} \{\omega(H) - 1\}$$

The tree-width of a class of graphs $\mathcal{C}$ is bounded if there is a constant k such that $tw(G) \leq k, \ \forall G \in \mathcal{C}$.

Theorem (Courcelle (1990))

Every graph property definable in the monadic second-order logic of graphs can be decided in linear time on graphs of bounded tree-width.*

Note: There are several other width parameters, namely: rank-width, clique-width, path-width, and they are bounded one to each other.

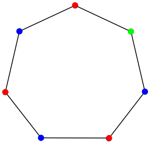
*Example: coloring, maximum independent set, maximum clique

Motivation of studying even-hole-free graphs

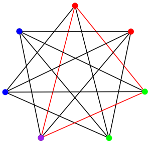
Relation to **perfect graphs**

- G is perfect if $\forall\, H \subseteq_{\text{ind}} G$, $\chi(H) = \omega(H)$ †
- Strong Perfect Graph Conjecture (by Berge, 1961; and proved by Chudnovsky, Robertson, Seymour, Thomas, 2002):

  G is perfect if and only if G is (odd hole, odd antihole)-free



odd hole



odd antihole

an antihole is *the "complement" of a hole*

† Every graph G satisfies $\chi(G) \geq \omega(G)$

Dichotomy between perfect graphs and EHF graphs

- Perfect graphs are **(odd hole, odd antihole)-free**
- EHF graphs are **(even hole, even antihole length ≥ 6)-free**

Comparison of the decomposition theorems

	EHF graphs [‡]	**Perfect graphs**[§]
Basic graphs	cliques, holes, long pyramids, nontrivial basic	bipartite, $\overline{\text{bipartite}}$, L(bipartite), $\overline{\text{L(bipartite)}}$ doubled graphs
separators	2-join, star separator	2-join, $\overline{\text{2-join}}$ homogeneous pair, balanced skew partition

Decomposition theorem: If G belongs to $\mathcal{C}$ then G is either "basic" or G has some particular separator.

[‡] Ref: Conforti, Cornuéjols, Kapoor, Vušković (2002)

[§] Ref: Chudnovsky, Robertson, Seymour, Thomas (2002)

Dichotomy between perfect graphs and EHF graphs

	EHF graphs	Perfect graphs
Structure	“simpler”	more complex
Maximum clique	poly-time	poly-time
Coloring	?	poly-time
Maximum independent set	?	poly-time

- Goal of study: to have better understanding of the structure of even-hole-free graphs

Part 2: Survey on tree-width of EHF graphs

Survey on EHF graphs having *bounded* tree-width

Remark: *in general, the tree-width of even-hole-free graphs is unbounded*

- *Planar* EHF $\rightarrow$ $tw \leq 49$ [Silva, da Silva, Sales (2010)]
- *K_3-free* EHF $\rightarrow$ $tw \leq 5$ [Cameron, da Silva, Huang, Vušković (2018)]
- *Pan-free* EHF $\rightarrow$ $tw \leq 1.5\omega(G) - 1$ [Cameron, Chaplick, Hoàng (2015)]
- *Cap-free* EHF $\rightarrow$ $tw \leq 6\omega(G) - 1$ [Cameron, da Silva, Huang, Vušković (2018)]

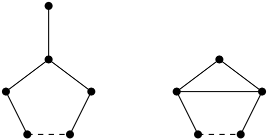
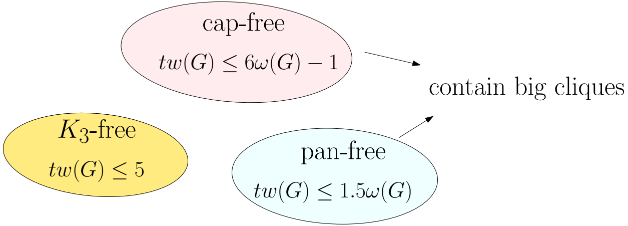


Figure: Pan and cap

Problem statement



Problem (Cameron, Chaplick, Hoàng (2018))

Let G be even-hole-free, is $tw(G) \leq f(\omega(G))$?

- No, we prove that even-hole-free graphs with no K_4 may have arbitrarily large tree-width

Part 3: Results

Truemper configurations

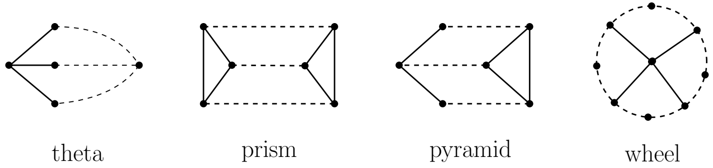


Figure: Truemper configurations; dashed lines represent paths of length at least 1

Truemper configurations

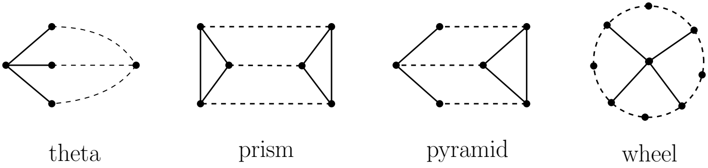
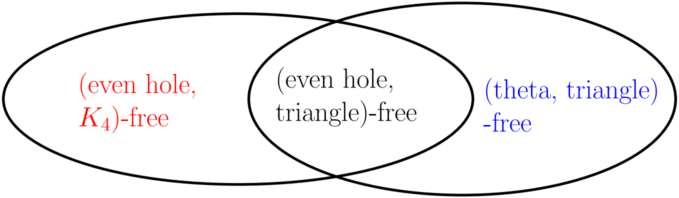


Figure: Truemper configurations; dashed lines represent paths of length at least 1

- They appear in the decomposition theorems of graphs in the classes

	EHF graphs	Perfect graphs
Theta	×	✓
Prism	×	✓
Pyramid	✓	×
Wheel	(no even wheel)	(no wheel of some kind)

Theta-free graphs (TTF) & even-hole-free graphs (EHF)



	(Even hole, K_4)-free graphs	(Theta, triangle)-free graphs
Theta	$\times$	$\times$
Prism	$\times$	$\times$
Pyramid	$\times$	$\checkmark$
Wheel	$\checkmark$	$\checkmark$

3.1. Layered Wheel

Summary of results

Layered wheels: family of graphs in the classes with high tree-width

Layered wheel G_ℓ

- $\ell \geq 1$ *is the number of "layers"*

Theorem (S., Trotignon (2019))

$\forall \ell \geq 1$ *integers,* $\exists$ *a theta-free graph* G_ℓ *s.t.*

$$\ell \leq tw(G_\ell) \leq c \cdot \log(|V(G_\ell)|), \text{ for some constant } c$$

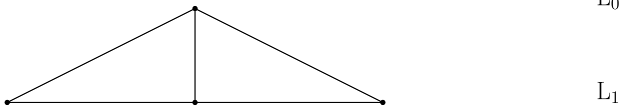
$\forall \ell \geq 1$ *integers,* $\exists$ *an (even hole,* K_4*, pyramid)-free graph* G_ℓ *s.t.*

$$\ell \leq tw(G_\ell) \leq c \cdot \log(|V(G_\ell)|), \text{ for some constant } c$$

Construction: *(theta, triangle)-free* layered wheel

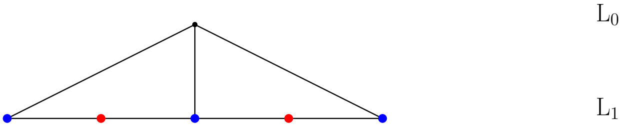
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



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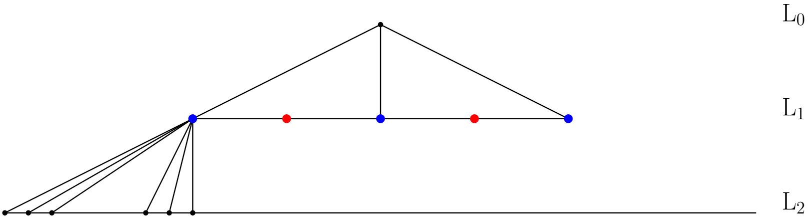

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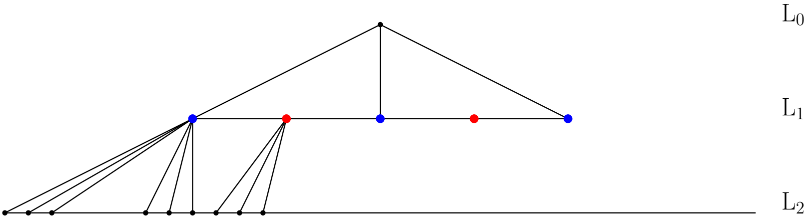
Construction: *(theta, triangle)-free* layered wheel



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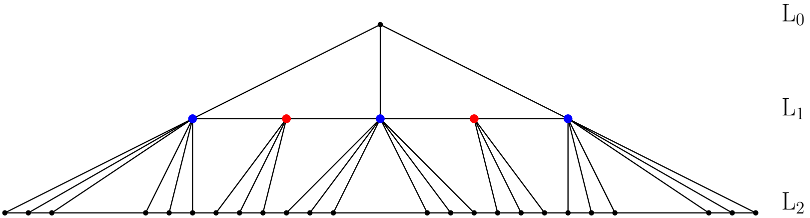

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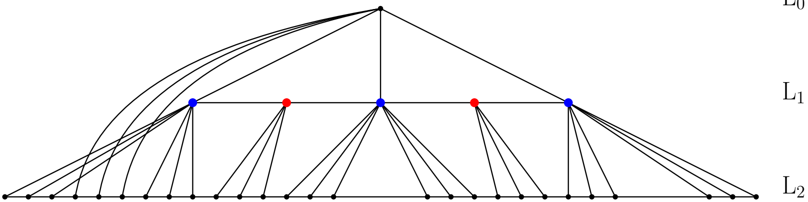

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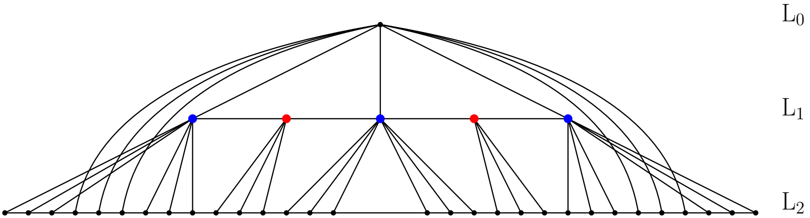
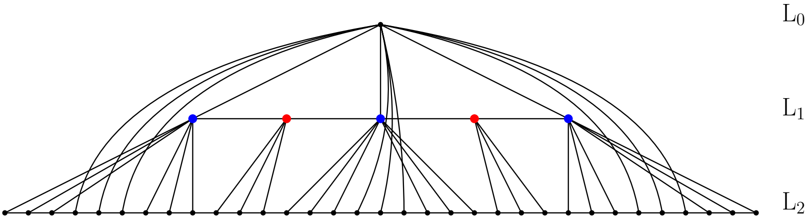


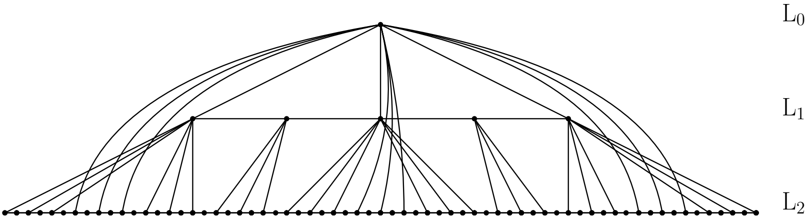
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



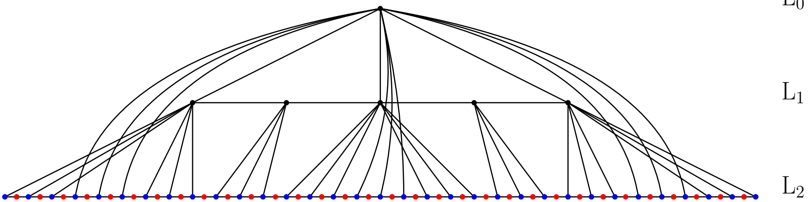
TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

Construction: *(theta, triangle)-free* layered wheel



TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

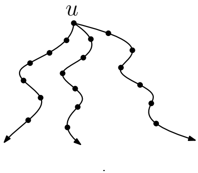
Construction: *(theta, triangle)-free* layered wheel

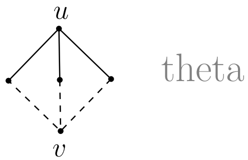


TTF layered wheel $G(\ell, k)$, with $\ell = 2$ and $k = 4$

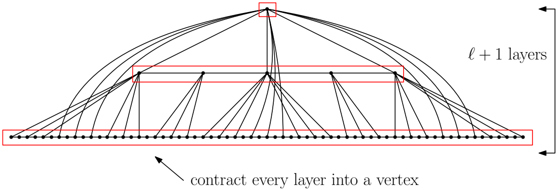
Sketch of proof

- $G_{\ell,k}$ is full of subdivision of claws, but it is theta-free





- $tw(G_{\ell,k}) \geq \ell$, because $G_{\ell,k}$ contains big clique minor



Logarithmic bound on the tree-width of layered wheels

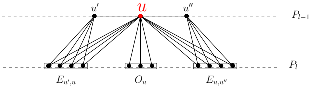
Theorem (S., Trotignon (2019))
$tw(G_{\ell,k}) = O(\log(|V(G_{\ell,k})|))$

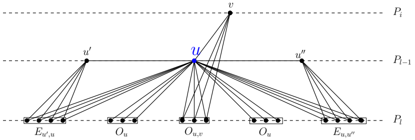
Key of proof:

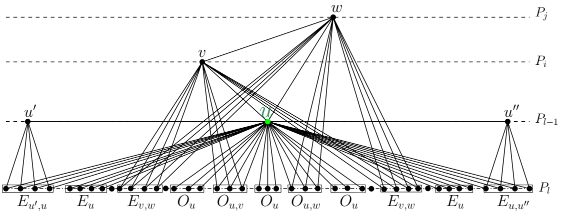
1. To reach $tw(G_{\ell,k}) \geq \ell$, it must be $|V(G_{\ell,k})| \geq 3^{\ell}$ vertices.
2. Upper bound: $tw(G_{\ell,k}) \leq 2\ell$.

Construction: *(even hole, K_4)-free* layered wheel

- The first two layers are similar to TTF-layered-wheel







3.2: Bounds on Tree-width

Motivation: the logarithmic conjecture (*now proved*)

Conjecture (*Logarithmic tree-width*; S., Trotignon (2019))

$\exists c$ constant s.t. $\forall$ (even hole, K_4)-free graph G,
$tw(G) \leq c \log |V(G)|$.

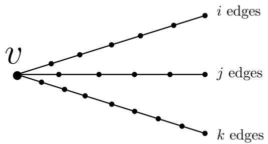
Theorem (Bodlaender (1988))

$\forall G$, given a tree decomposition of width w, the Weighted Maximum Independent Set can be solved in time $\mathcal{O}(2^w \cdot n)$.

Excluding $S_{i,j,k}$ & implication on tree-width

$\forall i, j, k, t \in \mathbb{Z}^T$, the following classes have bounded tree-width:

- (theta, triangle, $S_{i,j,k}$)-free graphs
- (even hole, pyramid, K_t, $S_{i,j,k}$)-free graphs



$S_{i,j,k}$

Why excluding $S_{i,j,k}$?

- Graphs with no subdivision of claw have been widely studied.

The logarithmic conjectures are proved for (theta, triangle)-free graphs

Induced subgraphs and tree decompositions III. Three-path-configurations and logarithmic treewidth, (T. Abrishami, M. Chudnovsky, S. Hajebi, S. Spirkl, 2021)

3.3. EHF Graphs of Bounded Maximum Degree

Motivation

Layered wheels contain

- large clique minor
- vertices with high degree

Are the two conditions necessary?

What is the tree-width of even-hole-free graphs, that:

1. have no big clique minor?
2. have bounded degree?

Even-hole-free graphs with maximum degree $\Delta \leq 3$

Theorem (*Decomposition;* Aboulker, Adler, Kim, S., Trotignon (2020))

Let G be a subcubic (theta, prism)-free graph.[¶] *Then one of the following holds:*

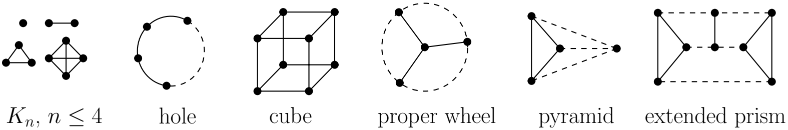
- *G is a basic graph;*
- *G has a clique separator of size at most 2;*
- *G has a proper separator.*

Conversely, every graph in the class can be built from a basic graph, by a sequence of *gluing along the separators*

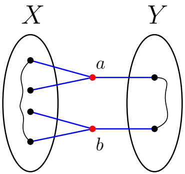
[¶] The theorem is proved for (theta, prism)-free graphs, which form a superclass of EHF graphs

Even-hole-free graphs with $\Delta \leq 3$

The basic graphs:



Proper separator:



Tree-width of subcubic EHF graphs

Theorem (*Tree-width;* Aboulker, Adler, Kim, S., Trotignon (2020))

Every **subcubic (theta, prism)-free graph** *(and therefore every even-hole-free subcubic graph) has tree-width at most 3.*

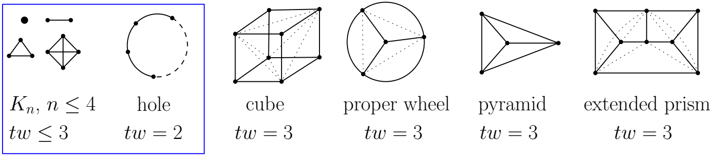


Figure: Chordal graphs containing the basic graphs

Tree-width of subcubic EHF graphs

- Gluing along a clique and proper gluing preserve the tree-width

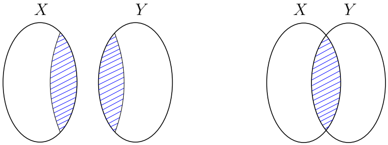


Figure: Gluing along a clique separator

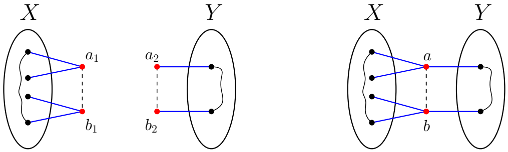


Figure: Gluing along a proper separator

Structure Theorem of EHF pyramid-free graphs $\Delta = 4$

Theorem (*Decomposition;* S., Trotignon (2020))

Let G be an **(even hole, pyramid)-free graph with** $\Delta(G) \leq 4$*. Then one of the following holds:*

- *G is a basic graph;*
- *G has a clique separator of size at most 3;*
- *G has a proper separator for* $\mathcal{C}$*.*

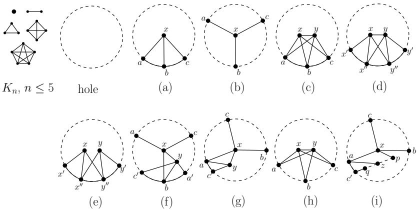


Figure: Basic graphs in the decomposition of the class

The tree-width of EHF pyramid-free graphs $\Delta = 4$

Theorem (*Tree-width;* S., Trotignon (2020))

Every (even hole, pyramid)-free graph with $\Delta \leq 4$ has tree-width at most 4.

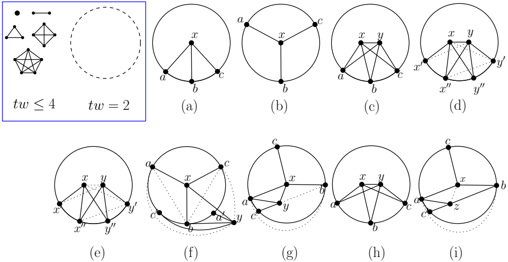


Figure: The basic graphs

- Gluing along a clique and proper gluing preserve the tree-width

Result on EHF graphs of bounded maximum degree

Induced subgraphs and tree decompositions I. Even-hole-free graphs of bounded degree (T. Abrishami, M Chudnovsky, K. Vušković, 2020)

Part 4: Closing

Publication

N. L. D. Sintiari and N. Trotignon.
(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 1 : Layered wheels
Published in Journal of Graph Theory (*CoRR*, abs/1906.10998), 2021.

M. Pilipczuk, S. Thomass, N. L. D. Sintiari, and N. Trotignon.
(Theta, triangle)-free and (even hole, K_4)-free graphs. Part 2 : Bounds on treewidth.
Published in Journal of Graph Theory (*CoRR*, abs/2001.01607), 2021.

P. Aboulker, I. Adler, E. J. Kim, N. L. D. Sintiari, and N. Trotignon.
On the tree-width of even-hole-free graphs.
Published in European Journal of Combinatorics (*CoRR*, abs/2008.05504), 2021.

Open problems

Conjecture (*now proved*)

$\exists f$ s.t. if $\mathrm{tw}(G) > f(k)$, then G contains (as induced subgraph):

- *a subdivision of a $(k \times k)$-wall; or*
- *the line graph of a subdivision of a $(k \times k)$-wall; or*
- *a vertex of degree at least k.*

Conjecture (*Grid-minor-like theorem (stronger version)*)

$\exists f$ s.t. if $\mathrm{tw}(G) > f(k)$, then G contains (as induced subgraph):

- *K_k, $K_{k,k}$; or*
- *a subdivision of a $(k \times k)$-wall; or*
- *the line graph of a subdivision of a $(k \times k)$-wall; or*
- *a wheel with at least k spokes.*

Open problems

A paper towards the conjecture:

Induced subgraphs and tree decompositions II. Toward walls and their line graphs in graphs of bounded degree (T. Abrishami, M Chudnovsky, C. Dibek, S. Hajebi, P. Rzażewski, s. Spirkl, K. Vušković, 2021)

Grid Induced Minor Theorem for graphs of small degree (T. Korhonen, 2022)

Thank you for listening!