TD n° 12 - Queues

Exercice 1. *Stationary analysis of M/M/1 queues*

We study the queue M/M/1 with one server and an infinite capacity. The inter-arrival times and the service times follow independent i.i.d. exponential laws of respective parameters $\lambda > 0$ and $\mu > 0$. We set $\rho = \frac{\lambda}{\mu}$ the utilization rate of the server. The number of clients in the queue N_t (including the one being served) is a continuous time Markov chain with values in $\mathbb{N}$.

1. Give the infinitesimal generator of the chain.

2. Provide a criterion of existence of an invariant distribution $(\pi_n)_{n\in\mathbb{N}}$ and compute it explicitely when it exists.

For the next questions, we assume that such a invariant distribution exists and the queue is in stationary mode (also called stationary regime / stationary state /steady-state / equilibrium), that is the law of N_0 is π and thus $\forall t \in \mathscr{R}_+$, the law of N_t is π. Conducting a stationary analysis of the queue consists in answering questions by choosing any t and computing performance measures with regard to the law of N_t that is π (probabilities will be denoted $\mathbb{P}_\pi$ and expectations $\mathbb{E}_\pi$).

3. Give the probability R that the server is busy.

4. Give the expectation L of the number of clients in the system.

5. Give the expectation L_q of the number of clients waiting in the queue (excluding the one being treated). Do we have $L_q = L - 1$? Why?

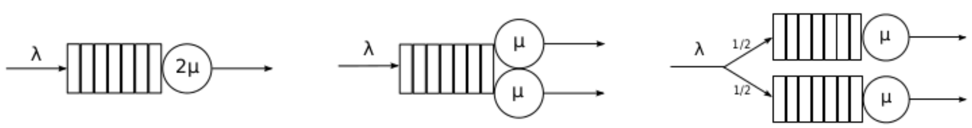
6. Suppose that the service policy is FIFO. What is the expectation W of the waiting time of a client (including its service time)?

7. Show that the output traffic is a Poisson process with intensity λ.

Exercice 2. *Comparing three small queuing systems*

We want to compare the three following systems :

1. Poisson process input traffic of parameter λ in one queue with a server with exponential service time of parameter 2μ.
2. Poisson process input traffic of parameter λ in one queue with two independent servers, each one with exponential service time of parameter μ.
3. Poisson process input traffic of parameter λ routed in one of two independent queues, with probability 1/2 for each queue and an exponential service of parameter μ for each queue.



1. Intuitively, which system performs the best?

2. Show that the arriving process of each queue in the third system is a Poisson process with rate $\lambda/2$.

3. For each system, show that it is a regular jump markov chain and describe the infinitesimal generator graph.

4. For each system, give the stability condition, i.e. the condition over λ and μ ensuring an invariant distribution.

5. For each system, compute the average number of clients in the system, assuming it is in the stationary state.

6. Where are the differences between the three models? Which one is the best?