TD n°8 (Markov Chain)

Exercice 1. *The first use of stopping theorem for martingales*

Theorem (Optional stopping theorem (Doob's Theorem)). *Let (M_n) be martingale (resp. sub-/super-) for (X_n) and T be stopping times for (X_n). If at least one of the following conditions holds :*

1. *$T \leq N$ a.s., where $N \in \mathbb{N}$*
2. *$T < \infty$ and $\forall n \in N$, $|M_n| \leq C$ a.s., where $C \in \mathbb{R}_+$*
3. *$\mathbb{E}(T) < \infty$ and $\forall n \in \mathbb{N}$, $|M_{n+1} - M_n| \leq C$ a.s., where $C \in \mathbb{R}_+$*

Then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ (resp. $\geq$ / $\leq$)

The first application : let (X_n) be symmetric walk on $\mathbb{Z}$, $0 \leq i \leq N$, let $T = \tau_{[0,N]}$ be time absorbed by 0 or N. Propose the martingales to calculate the following values :

— The probability of absorption $\mathbb{E}_i(T)$ starting from i, i.e. $\mathbb{P}_i(T_N < +\infty)$,
— The mean of absorption $\mathbb{E}_i(T)$ starting from i.

Exercice 2. *Foster theorems*

Given the following theorems

Theorem (First Foster theorem). *Let (X_n) be a homogeneous irreducible Markov chain of general term $p_{i,j}$ on a countable set E. If there exists a function $h : E \to \mathbb{R}^+$, a finite set F and a constant $\varepsilon > 0$ such that :*

$$\sum_{k\in E} p_{ik} h(k) < \infty \quad \textit{for all } i \in F$$
$$\sum_{k\in E} p_{ik} h(k) \leq h(i) - \varepsilon \quad \textit{for all } i \notin F,$$

then (X_n) is positive recurrent.

Theorem (Second Foster theorem). *Let (X_n) be a homogeneous irreducible Markov chain of general term $p_{i,j}$ on a countable set E. If there exists a function $h : E \to \mathbb{R}^+$ and a finite set F such that :*

$$\mathbb{E}(h(X_1) - h(X_0) | X_0 = i) < +\infty \quad \forall i \notin F$$
$$h(j_0) > \max_{i\in F} h(i) \quad \textit{for some } j_0 \notin F$$
$$\sum_{k\in E} p_{ik} h(k) \geq h(i) \quad \textit{for all } i \notin F$$

then (X_n) is not positive recurrent.

Consider the following random walk on $\mathbb{N}$: if $X_n = 0$ then $X_{n+1} = 1$ with probability 1, and if $X_n \geq 1$, then

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob. } p \\ X_n - 1 & \text{with prob. } 1-p \end{cases}$$

Using Foster theorems, determine for which values of p this Markov chain is positive recurrent.

Exercice 3. *Aloha Stabilization*

Aloha is a communication protocol on a canal shared by several stations unaware of each other. Transmissions and retransmissions can only start at times of type $k\Delta$ with k integer and $\Delta > 0$ the width of a *slot*. When two stations try to transmit simultaneously messages, they interfere and none is actually transmitted. These *conflicts* are detected by stations. The protocol is the following :

— Fresh messages systematically try to pass right after their arrival.

— In case of conflict, each concerned station independently tries to retransmit its message at the next slot with probability $0 < \nu < 1$.

We denote by A_n the number of fresh messages arrived at the beginning of slot n and X_n the number of messages delayed at slot n. We assume that the r.v. A_n are i.i.d. and we set $a_i = \mathbb{P}(A_n = i)$, $\lambda = \mathbb{E}(A_n) = \sum_{i=0}^{\infty} i\, a_i$.

A - Aloha Instability :

1. Give the probability $b_i(k)$ that i stations try to retransmit if k stations are in conflict.

We will assume to simplify that the retransmission of a message depends only on itself and not on its station. This has the weird consequence that two messages from the same station can conflict. Under this assumption, $b_i(k)$ represents the probability that i messages are retransmitted if k ones are delayed.

2. Give the probability $p_{k,l}$ to pass from k to l delayed messages.

3. Show that this protocol is unstable (i.e., (X_n) is not positive recurrent).

4. What does it actually means for the protocol?

B - Aloha Stabilization :

Instead of using a retransmission policy with ν fixed, we will try to reach stability using $\nu(k)$ depending on the number of delayed messages. We will show that the following condition implies stability.

$$\lambda < \liminf_{k \to +\infty} (b_1(k) a_0 + b_0(k) a_1)$$

It is equivalent to the existence of $\varepsilon > 0$ and a finite set $F \subset \mathbb{N}$ such that

$$\lambda < b_1(k) a_0 + b_0(k) a_1 - \varepsilon \quad \text{for all } k \notin F.$$

5. Under this assumption, prove the stability of the protocol.

6. Study the extrema of $g_k(\nu) = (1-\nu)^k a_1 + k\nu(1-\nu)^{k-1} a_0$.

7. Noticing that $\left(\frac{k-1}{k - a_1/a_0}\right)^{k-1} \xrightarrow{k\to\infty} \exp(\frac{a_1}{a_0} - 1)$, give a sufficient stability condition.

8. Explicit this condition when A_n follows a Poisson distribution.

9. What is the drawback of this policy?