TD n°7 (Markov Chain)

Exercice 1. *Bursty traffic model*

To model data arrivals (or error sequences) in a system, rather than choosing i.i.d Bernoulli random variables, one can use a markovian model with two states : state 0 (no arrival / no error) and state 1 (data arrival / error). Maximal sequences of 1 (resp. 0) are called bursts (resp. silences).

1. Let $t \in \mathbb{N}$, denote $X(t)$ the random variable equal to 1 in a burst at time t and 0 otherwise. Suppose that the process $X(t)$ is markovian with $\mathbb{P}(X(t+1) = 1|X(t) = 0) = p$ and $\mathbb{P}(X(t+1) = 0|X(t) = 1) = q$. For the stationnary regime, what is the law for the length of bursts (resp. silences)?

2. The IBP model (Interrupted Bernoulli Process) is a variant where the process alternates between silences and bursts. All time lapses are assumed independent and following geometric laws of parameter p and q. During silence periods, no data/error arrives, and during burst periods, at each time step, some new data (or error) may occur independently with probability α? Let $X(t)$ the random variable equal to 1 (resp 0) if time t belong to a burst (resp. a silence). Let $Y(t)$ be the integer random variable equal to 1 if a new data arrives, and 0 otherwise. Is $Y(t)$ markovian? If not, is it possible to add some information to get a markovian process.

Exercice 2. *Stochastic process with bounded dependence*

Let $(X_t)_{t\in\mathbb{N}}$ be a stochastic process on E and such that there exists $\ell \in \mathbb{N}$, $\forall t > \ell$, $\forall i_{t+1}, i_t, \dots, i_0 \in E$,

$$\mathbb{P}(X_{t+1} = i_{t+1}|X_t = i_t, X_{t-1} = i_{t-1}, \dots, X_0 = i_0) = \mathbb{P}(X_{t+1} = i_{t+1}|X_t = i_t, X_{t-1} = i_{t-1}, \dots, X_{t-\ell} = i_{t-\ell})$$

Show how to reduce its study to classical Markov chains.

Exercice 3. *Contention resolution in the IEEE 802.11 Protocol in saturation (continue of last week)*

Consider a system with N stations emitting wifi messages. Time is discrete and an integer $W > 0$ is fixed. At the MAC level, the communication protocol follows the following rules :

- If a station i wishes to transmit a message, it independently draws a random integer W_i uniformly in $\{0, 1, \dots, W-1\}$. At each time step, it decreases this integer by 1. When W_i reaches 0, it emits its message at once.
- When a station emits a message alone during the time slot, then the message is perfectly transmitted. If the station has another message to transmit, it draws a new random integer at the next time slot and the same process restarts.
- If two or more stations emits in the same time slot, the messages interfere, it is a *collision*. All those messages fail their transmissions. Then each station tries to resend its message by drawing a new independent random integer at the next time slot and the same process restarts.

1. We focus on the *saturated mode* where each station has an infinite number of messages to transmit. How will the system evolve?

2. Does the Markov Chain correspond to the system admit a stationary distribution? Compute it if it does!

3. In the saturated mode, you observe a trajectory of the system and the *individual throughput rate* of a station along this trajectory, i.e. given time t, the number of messages fully transmitted by this station between time 0 and time $t-1$, divided by t. How will this rate evolve along the trajectory?

4. The N stations are in saturated mode and you want to optimize the *global throughput rate* of the system, which measures the total number of fully transmitted messages in the system per time unit. How would you choose W? Explain your choice (start for instance by defining mathematically the global throughput rate).

Exercice 4. *Chapman-Kolmogorov without Markov*

Let $(X_{2n+1})_{n\in\mathbb{N}}$ be i.i.d. random variables with values in $\{-1,1\}$ and $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. Define also for all $n \in \mathbb{N}^*$, $X_{2n} = X_{2n-1}X_{2n+1}$.

1. Check that the random variables $(X_{2n})_{n\in\mathbb{N}^*}$ are independent and follow the law of X_1. Show that X_n et X_{n+1} are independent. Deduce from this that $(X_n)_{n\in\mathbb{N}^*}$ are independent.

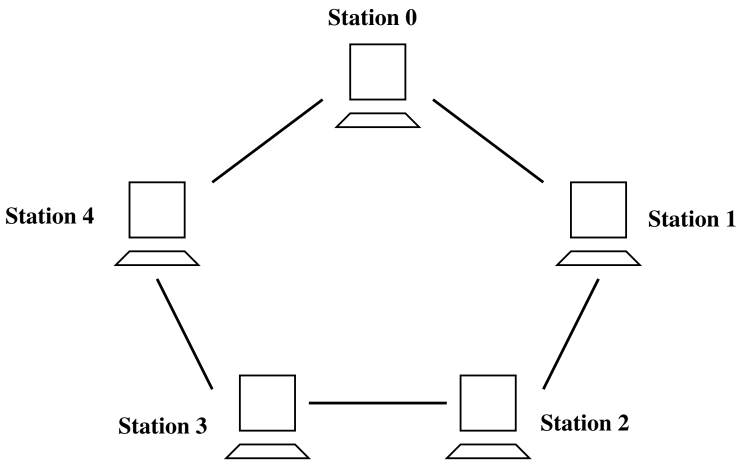
2. Compute $\mathbb{P}(X_{m+n} = j|X_m = i)$ for all $m, n \in \mathbb{N}^*$ et $i, j \in \{-1,1\}$. Deduce the Chapman-Kolmogorov property.

3. Compute $\mathbb{P}(X_{2n+1} = 1|X_{2n} = -1, X_{2n-1} = 1)$. Deduce that (X_n) is not a Markov chain.

4. Consider the couples $Z_n = (X_n, X_{n+1})$, $n \in \mathbb{N}^*$. Is it a Markov chain? Is it homogeneous?

Exercice 5. *Herman's Algorithm*

Consider N stations around a ring network, where N is odd, as illustrated below for $N = 5$. Each station can be either in state 0 or state 1. Time is discrete and at each time step, all stations perform simultaneously the following algorithm. Let x_i be the state of station i, then if $x_i \neq x_{i-1 \mod N}$, its new state becomes $x'_i = x_{i-1}$, and if $x_i = x_{i-1 \mod N}$, its new state becomes $x'_i = 1 - x_i$ with probability $1/2$ or $x'_i = x_i$ with probability $1/2$.



How does evolve the set of all states on the ring? What kind of stochastic process is it? Describe all interesting features of this process. What is its asymptotic behaviour? Give as many details as you can or at least list the questions that could be interesting to study.