TD n°6 (Markov Chain)

Exercice 1. *Markovian or not ?*

1. A dice is thrown repeatedly. For each case, find out whether the sequence of random variables is a Markov chain, and then whether it is homogeneous by giving its transition matrix :
(a) M_n is the largest value that occured during the first n drawings.
(b) N_n is the number of times the face 1 appeared during the first n drawings.
(c) For the n-th drawing, L_n is the time spent from the last occurence of face 1.
(d) For the n-th drawing, W_n is the waiting time spent until the next occurence of face 1.

2. Let $(X_n)_{n\in\mathbb{N}}$ be a Markov Chain over E and h a function from E to F. If h is injective, is $Y_n = h(X_n)$ still a Markov chain over F ? Same question if h is not necessarily injective ?

Exercice 2. *IEEE 802.11 Protocol to prevent collisions*

Consider a system with n stations emitting wifi messages. Time is discrete and an integer $W > 0$ is fixed. At the MAC level, the communication protocol follows the following rules :

- If a station i wishes to transmit a message, it independently draws a random integer W_i uniformly in $\{0, 1, \ldots, W-1\}$. At each time step, it decreases this integer by 1. When W_i reaches 0, it emits its message.
- When a station emits a message, if it is the only one emitting, then the message is perfectly transmitted. If the station has another message to transmit, it instantaneously draws a new random integer and the same process restarts.
- If two or more stations emits in the same time slot, the messages interfer, it is a *collision*. All those messages fail their transmissions. Each station tries to resend its message by drawing a new independent random integer and the same process restarts.

1. What is happening in the system if $W = 1$?

2. Suppose that at time $t = 0$, each station has a fixed number of messages, finite or infinite, to transmit one after the other. Show that the dynamics of the system can be described as a Markov chain $(X_t)_{t\in\mathbb{N}}$ over appropriate states. Illustrate this by drawing the transition graph when $n = 2$, $W = 2$ and each station has only 1 message to transmit. Same question when $n = 2$, $W = 3$ and each station has an infinite number of messages to transmit.

3. In which cases the chain you have constructed is irreducible and aperiodic ?

Exercice 3. *Beware of the Markov property*

1. The Markov property does not say that the past and the future are independent given *any* information about the present. Find a simple example of homogeneous markov chain $(X_t)_{t\in\mathbb{N}}$ over $E = \{1, 2, 3, 4, 5, 6\}$ such that

$$\mathbb{P}(X_2 = 6 | X_1 \in \{3, 4\}, X_0 = 2) \neq \mathbb{P}(X_2 = 6 | X_1 \in \{3, 4\})$$

2. The strong Markov property does not apply to any time random variable. Find an example of homogeneous markov chain $(X_t)_{t\in\mathbb{N}}$ and a time random variable T such that none of the two conditions of the strong markov properties are satisfied.

Exercice 4. *Casino shopping not gambling*

A cashier of your Casino store has already 10 people waiting when he arrives. He needs exactly 1 min to process with one client. But during this minute and independently from anything else, there is a probability

1/3 (resp. 1/6) that one new client (resp. two clients) join the queue. What is the average time for the cashier to empty the queue?

Exercice 5. *Bursty traffic model*

To model data arrivals (or error sequences) in a system, rather than choosing i.i.d Bernoulli random variables, one can use a markovian model with two states : state 0 (no arrival / no error) and state 1 (data arrival / error). Maximal sequences of 1 (resp. 0) are called bursts (resp. silences).

1. Let $t \in \mathbb{N}$, denote $X(t)$ the random variable equal to 1 in a burst at time t and 0 otherwise. Suppose that the process $X(t)$ is markovian with $\mathbb{P}(X(t+1)=1|X(t)=0)=p$ and $\mathbb{P}(X(t+1)=0|X(t)=1)=q$. For the stationnary regime, what is the law for the length of bursts (resp. silences)?

2. The IBP model (Interrupted Bernoulli Process) is a variant where the process alternates between silences and bursts. All time lapses are assumed independent and following geometric laws of parameter p and q. During silence periods, no data/error arrives, and during burst periods, at each time step, some new data (or error) may occur independently with probability α? Let $X(t)$ the random variable equal to 1 (resp 0) if time t belong to a burst (resp. a silence). Let $Y(t)$ be the integer random variable equal to 1 si a new data arrives, and 0 otherwise. Is $Y(t)$ markovian? If not, is it possible to add some information to get a markovian process.

Exercice 6. *Chapman-Kolmogorov without Markov*

Let $(X_{2n+1})_{n\in\mathbb{N}}$ be i.i.d. random variables with values in $\{-1,1\}$ and $\mathbb{P}(X_1=1)=\mathbb{P}(X_1=-1)=1/2$. Define also for all $n\in\mathbb{N}^*$, $X_{2n}=X_{2n-1}X_{2n+1}$.

1. Check that the random variables $(X_{2n})_{n\in\mathbb{N}^*}$ are independent and follow the law of X_1. Show that X_n et X_{n+1} are independent. Deduce from this that $(X_n)_{n\in\mathbb{N}^*}$ are independent.

2. Compute $\mathbb{P}(X_{m+n}=j|X_m=i)$ for all $m,n\in\mathbb{N}^*$ et $i,j\in\{-1,1\}$. Deduce the Chapman-Kolmogorov property.

3. Compute $\mathbb{P}(X_{2n+1}=1|X_{2n}=-1,X_{2n-1}=1)$. Deduce that (X_n) is not a Markov chain.

4. Consider the couples $Z_n=(X_n,X_{n+1})$, $n\in\mathbb{N}^*$. Is it a Markov chain? Is it homogeneous?