TD7 - Duality of Linear Programming

Indication of hardness: from (*) to (****).

1 Totally unimodular matrices

Exercise 1 - Basic facts on total unimodularity (*)

(a) Show that the transpose of a totally unimodular matrix is totally unimodular.

(b) Show that if M is totally unimodular then $\begin{pmatrix} M & I \end{pmatrix}$ is totally unimodular.

(c) Show that a submatrix of a totally unimodular matrix is unimodular.

Exercise 2 - Permutation matrix (*)

A $0-1$-quare matrix is a permutation matrix if exactly one coefficient on each row and on each column equals 1 and all the other coefficients equal 0.
Show that a permutation matrix is totally unimodular.

Exercise 3 - Discrepency (***)

Let A be an integer matrix. The discrepancy of A is the minimum d for which there exists a vector y with all its coefficients in $\{-1, 1\}$ such that (**1** denotes the vector of 1):

$$-d \cdot \mathbf{1} \leq Ay \leq d \cdot \mathbf{1}$$

(a) Give an upper bound on the discrepancy of any matrix.

(b) Show that if a matrix M is totally unimodular, then its discrepancy is at most 1.

(c) Show that if all submatrices of M have discrepancy at most 1, then M is totally unimodular.

Exercise 4 - Total unimodularity and integral solutions (**)

(a) Suppose A is totally unimodular matrix. Let b be any integer vector. Then every basic feasible solution of $P = \{x : Ax \leq b\}$ has integer coefficients.

(b) Suppose A is totally unimodular matrix. Let b, c be two integer vectors. Then every basic feasible solution of $P = \{x : Ax \leq b, 0 \leq x \leq c\}$ has integer coefficients.

Exercise 5 - Characterization of incidence matrices that are TU (*)

(a) Prove that the incidence matrix of an odd cycle is not TU.

(b) Dedude that a matrix with exactly two 1s on each row is TU iff there is a partition I_1, I_2 of the columns such that each row has exactly one 1 on the set of columns of I_1 and exactly one 1 on the set of columns of I_2.

2 *Max-Flow Min-Cut*

Exercise 6 - MaxFlow-MinCut

We are given a *directed graph* $D = (V, A)$, where $V = \{1, \ldots, N, s, t\}$ is the set of *nodes* (or *vertices* but we will prefer node to avoid confusion) and A is the set of directed arcs (an arc is an ordered pair of vertices). Let s, t be two special nodes where s is called the *source* and the node t is called the *sink* of the graph. We assume that there are no arcs going into the source s, and there are no arcs coming out of the sink t. For every arc (i, j) (*i.e.* arc from i to j), we are given also a capacity $c_{ij} \geq 0$. Suppose that we want to send units of 'flow' (representing e.g. water or data packets) along the arcs (in the direction of the arc) from node to node, injecting flow into the network at s (the source) and taking flow out of the network at t (the sink). We assume that there is no leakage in the system, so that the amount of flow that goes into an vertex equals the amount of flow that goes out of it and. Also, the amount of flow on arc (i, j) can be at most its capacity c_{ij}.

(a) Let x_a be an variable for each arc a. Prove that the following LP characterizes max-flow:

$$\begin{aligned} & \max \sum_{a=(s,u)} x_a \\ \text{subject to} \quad & \sum_{v/uv \in A} x(uv) = \sum_{v/vu \in A} x(vu) \text{ for all } u \in V \setminus \{s, t\} \\ & x_a \leq c_a \quad \text{for all } a \in A \\ & x_a \geq 0 \quad \text{for all } a \in A \end{aligned}$$

(b) Prove that the constraint matrix of this LP is TU. Does it mean that it has integral solutions?

(c) Prove that the following LP is the ILP of a minimum st-cut:

$$\begin{aligned} z^* = & \min \sum_{a \in A} c_a y_a \\ \text{subject to} \quad & y_u - y_v - y_a \leq 0 \text{ for every } a = (u, v) \\ & y_s = 1 \\ & y_t = 0 \\ & y_u, y_{uv} \in \{0, 1\} \end{aligned}$$

(d) Prove that the dual of the Max-Flow LP is the fractional relaxation of the ILP min st-cut.

(e) Conclude.

Exercise 7 - Applications of Max-Flow Min-Cut theorem (*)
Let $D = (V, A)$ be a (unweighted) directed graph. A *in-neighbor* (resp. out-neighbor) of v is a vertex u such that uv is an arc (resp. vu is an arc).

(a) Show that if there is an ordering of V such that s is the first vertex, t is the last vertex and every vertex v has an in-neighbor before it in the order and an out-neighbor after it in the order then there is a path from s to t passing through v.

(b) Let s, s', t, t' be four vertices of D. We want to count the flow from $\{s, s'\}$ to $\{t, t'\}$ (*i.e.* the number of paths starting in $\{s, s'\}$ and ending in $\{t, t'\}$). Free to transform D can you express this problem with a MaxFlow problem?

(c) A directed circuit is a path which starts and ends at the same point. Two circuits are arc-disjoint if no arc of one is included in the other. Let $v \in V(D)$. Free to introduce a new graph D' can you show that the number of arc-disjoint directed circuit containing v can be expressed as max-flow problem in D'.

3 *Other applications of TU matrices*

Exercise 8 - Transportation problem
Consider the following transportation problem: F is a set of warehouses that are owned by our company, G is a set of different goods and C is a set of clients (all sets are finite). Let $s_{ij} \geq 0$ be the amount of good $i \in G$, that is available in warehouse $j \in F$. Furthermore $d_{ik} \geq 0$ denotes the amount of good $i \in G$, that client $k \in C$ requests. It costs $c_{ijk} \geq 0$ to transport one unit of good $i \in G$ from warehouse j to customer $k \in C$.

All quantities are integer. (We assume that the costs grow linear with the amount and goods are splittable in integer quantities).

(a) Formulate an integer program that determines the cheapest way to transport the goods to the clients such that: The demand of each client is satisfied and the supplies of the warehouses are not exceeded.

(b) What can you say about the constraint matrix?

(c) Conclude.