# TD6 - Sensitivity analysis

*Indication of hardness: from (*) to (****).*

## 1 Sensitivity analysis and Duality

**Exercise 1 - Sensitivity Analysis (*)**
Solve the following LPs with the Simplex and determine on the optimal tableau:
(a) the optimal solution;
(b) the shadow prices on the three constraints;
(c) the range on the availability of each resource for which the basis of the optimal solution remains the same.

$$\begin{aligned} &\max x_1 + x_2 \\ &\text{suject to} \\ &x_1 + x_4 + x_5 - x_6 = 5 \\ &x_2 + 2x_4 - 3x_5 + x_6 = 3 \\ &x_3 - x_4 + 2x_5 - x_6 = -1 \end{aligned}$$

$$\begin{aligned} &\max 2x_1 - x_2 \\ &\text{subject to} \\ &x_1 + x_2 \geq 5 \\ &6x_2 + x_1 \leq 11 \\ &3x_1 - x_2 \leq 18 \end{aligned}$$

**Exercice 2 - Primal and dual solutions. (**)**
Consider the following LP:

$$\begin{aligned} z = \max c^t x \\ \text{suject to} \\ Ax \leq b \\ x \geq 0 \end{aligned}$$

Let $x^*, y^*$ and $z^*$ be respectively optimal solutions of the primal, dual and the optimal value. Prove that

$$z^* = (y^*)^t A x^*$$

## 2 Discrete Mathematics

**Exercise 3 - Vertex Cover (*)**
A *vertex cover* of a graph $G$ is a subset of vertices $X$ such that every edge of $G$ contains at least one vertex of $X$. We have seen during the first lecture a $\frac{1}{2}$-approximation of the minimum vertex cover problem. Here is another (more complicated) proof of this result using Linear Programming.
(a) Formulate the Minimun Vertex Cover (MVC) problem as an Integral LP.
(b) Write down the (Integral) dual of the MVC problem. Which problem is it?
(c) Using duality theorem, (re)show that there exists a 2-approximation algorithm of the MVC problem.

**Exercise 4 - Another approximation algorithm (***)**
We have seen during the first lecture a $\frac{1}{2}$-approximation of the minimum vertex cover problem. Here is another (more complicated) proof of this result using Linear Programming. Let us denote by MVC* the corresponding LP in real numbers. Let $x^*$ be an optimal solution of MVC* of value $v^*$. It can be interpreted as a weight function on the vertices (why?). A vertex is *saturated* for $x^*$ if the sum of the weights incident to this vertex equal $1$.

(a) Prove that for every edge $e = (u, v)$, $u$ or $v$ are saturated.

(b) Prove that all the basic feasible solutions are such that all the variables have values in $\{0, \frac{1}{2}, 1\}$.
(Show that you can algorithmically transform $x^*$ into that solution).

(c) Deduce a 2-approximation algorithm for Vertex Cover.

**Exercise 5 - Shortest Path (**)**
The shortest s-t-path problem is defined as follows. Given a graph $G = (V, E)$ with non-negative edge lengths and two designated vertices, $s$ and $t$, find the minimum length path from $s$ to $t$.

(a) Write an Integer Linear Program for the shortest $s - t$-path problem.

(b) Prove that the optimal solution of the fractional relaxation of the shortest path problem has the same value as the integral problem.