TD5 - Duality

Indication of hardness: from (*) to (****).

1 Simplex algorithm (again)

Exercise 1 - One cannot improve the solution. (**)

- Given an example of tableau of LP where the algorithm cannot improve the current solution for any possible choice of pivot with respect to the current solution.
- Give a geometrical example (on two dimensions) where the same condition is satisfied.

Exercise 2 - A last simplex algorithm (**)
Use the Simplex Algorithm to show that the following problem is unbounded:

$$\begin{aligned} \max z = -x_1 + 2x_2 + x_3 & \\ \text{subject to:} & \\ 3x_1 + x_2 - 4x_3 &\leq 4 \\ x_1 - x_2 - x_3 &\leq 10 \\ x_1 - 2x_2 + 6x_3 &\leq 9 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

2 Duality

Exercise 3 - Computing duals (*)
Give the dual of the following LPs:

$$\begin{aligned} \max 2x_1 + 3x_2 + x_3 & \\ \text{subject to} & \\ x_1 + x_2 + x_3 &\leq 40 \\ 2x_1 + x_2 - x_3 &\geq 10 \\ -x_2 + x_3 &\geq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

$$\begin{aligned} \max 1000x_1 + 1200x_2 & \\ \text{subject to} & \\ 10x_1 + 5x_2 &\leq 200 \\ 2x_1 + 3x_2 &= 60 \\ x_1 &\leq 12 \\ x_2 &\geq 6 \\ x_1, x_2 &\geq 0 \end{aligned}$$

$$\begin{aligned} \max x_1 + 5x_2 - x_3 & \\ \text{subject to} & \\ x_1 + 5x_2 + 3x_3 &= 12 \\ x_1 - x_3 &\leq 5 \\ x_2 - 5x_3 &\geq 1 \\ x_1 &\geq 0 \end{aligned}$$

$$\begin{aligned} \max x_1 & \\ \text{subject to} & \\ -5x_1 + 3x_2 &= 200 \\ 11x_1 + 3x_2 &= 60 \\ x_2 &\geq 6 \\ x_1 &\geq 0 \end{aligned}$$

Exercise 4 - Determining the dual (*)

(a) Show that the dual of a non-positive variable is a $\leq$ constraint.

(b) Show that the dual of a free variable is an equality constraint.

(c) Show that the dual of a $\geq$ constraint is a non positive variable.

Exercise 5 - Unboundedness and feasibility (*)

(a) Find a linear program (P) such that both (P) and its dual (D) are not feasible.

(b) Find a linear program (P) such that (P) is not feasible and (D) is unbounded.

Exercise 6 - Self duality (**)
Consider the LP (P)

$$\begin{aligned} \min q^t z & \\ \text{subject to} & \\ Mz &\geq -q \\ z &\geq 0 \end{aligned}$$

in which the matrix M is *skew symmetric*; i.e. $M = -M^t$.

(a) Prove that the dual of this problem is itself.

(b) Show that (P) is feasible iff its optimal value is bounded.

Certificate of optimality

Exercise 7 - Certificate of optimality

(a)

$$\begin{aligned} \max x_1 + x_2 & \\ \text{subject to} & \\ x_1 + 2x_2 &\leq 4 \\ 4x_1 + 2x_2 &\geq 12 \\ -x_1 + x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(b)

$$\begin{aligned} \max x_1 + 7x_2 + 3x_3 & \\ \text{subject to} & \\ -x_1 + 3x_2 - 2x_3 &\leq 0 \\ x_1 - 4x_2 + 2x_3 &\leq 0 \\ x_2 + 2x_2 + 3x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Compute the dual of these LP and determine if:

- Are the points $(2/3, 5/3)$, $(3, 0)$ or $(8/3, 2/3)$ optimal for (a)?
- Is the point $(3, 1, 0)$ optimal for (b)?

3 *Dualite in Discrete Mathematics*

Exercise 8 - Dominating Set

A *dominating set* is a subset X of vertices of G such that for every v in V, then $v \in X$ or v is incident to a vertex of X.

(a) Formulate the Minimum Dominating Set (MDS) problem as an ILP.

(b) Prove that the integrality gap cannot be bounded by any constant (i.e. the ratio between the optimal value and the fractional optimal value can be larger than any constant).

(c) What is the dual of the MDS problem? Interpret it as a "generalization" of an independent set problem.