TUTORIAL IX

Midterm preparation

Problem 1 (Basics)**.** For each one of these statements, say whether it is true or false and provide a brief justification.

1. Define the distribution $P_X = (1/5, 1/5, 1/5, 2/5)$. We have $H(X) = \log_2 5$.
2. For any random variable $X \in \mathcal{X}$ and any $x \in \mathcal{X}$, we have $P_X(x) \leq 2^{-H(X)}$.
3. Define the channel W with binary input and output given by $W(0|0) = 1/3, W(1|0) = 2/3, W(0|1) = 1/3, W(1|1) = 2/3$. The capacity of this channel is 0.
4. Define the tripartite mutual information $I(X:Y:Z) := I(X:Y) - I(X:Y|Z)$. For any random variables X, Y, Z, we have $I(X:Y:Z) \geq 0$.
5. For any random variables X_1, X_2, we have $H(X_1X_2) = H(X_1) + H(X_2)$.
6. Consider the distribution $P_X = (1/2, 1/4, 1/8, 1/16, 1/16)$. The code with the shortest expected length for this source has expected length exactly $H(X)$.
7. Let $X_1, \dots, X_n$ be iid random variables each living in the finite set $\mathcal{X}$. A sequence $x^n = (x_1, \dots, x_n) \in \mathcal{X}^n$ is said to be ϵ-typical if $2^{-n(H(X_1)+\epsilon)} \leq P_{X_1 \dots X_n}(x_1 \dots x_n) \leq 2^{-n(H(X_1)-\epsilon)}$. Now a sequence $x^n = (x_1, \dots, x_n)$ is said to be ϵ-*strongly* typical if $(1-\epsilon)P_{X_1}(a) \leq \frac{N(a|x^n)}{n} \leq (1+\epsilon)P_{X_1}(a)$ for all $a \in \mathcal{X}$. Here $N(a|x^n)$ denotes the number of times the symbol a occurs in the sequence x^n.

   The statement is that if x^n is ϵ-strongly typical, then x^n is $c \cdot \epsilon$-typical where c is a constant that is independent of n but can depend on the distribution P_{X_1}.
8. If x^n is ϵ-typical, then it is also $c \cdot \epsilon$-strongly typical for a constant c that is independent of n but can depend on the distribution P_{X_1}.

Problem 2 (Capacity of a simple channel)**.** Define the channel W with binary input $\mathcal{X} = \{0, 1\}$ and binary output $\mathcal{Y} = \{0, 1\}$ and $W(0|0) = 1$, $W(0|1) = \frac{1}{2}$ and $W(1|1) = \frac{1}{2}$. Show that the information capacity $\mathrm{C}(W) = \sup_{x \in [0,1/2]} h_2(x) - 2x$, where $h_2(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function.

Problem 3 (Compression with side information)**.** In class, we showed that in order to compress a source $X \in \mathcal{X}$ with distribution P_X into ℓ bits, the minimum error probability $\delta^{\text{opt}}(P_X, \ell)$ satisfies for any $\tau > 0$,

$$\mathbf{P}\left\{\log_2 \frac{1}{P_X(X)} > \ell + \tau\right\} - 2^{-\tau} \leq \delta^{\text{opt}}(P_X, \ell) \leq \mathbf{P}\left\{\log_2 \frac{1}{P_X(X)} > \ell\right\} . \tag{1}$$

[Added remark: We did not do it this year, but in the tutorial, you proved something very similar]

As a consequence, we showed that in the case where the source X^n is n independent copies $X_1, \dots, X_n$ of X, then

$$\lim_{n\to\infty} \delta^{\text{opt}}(P_{X^n}, Rn) = \begin{cases} 1 & \text{if } R < H(X) \\ 0 & \text{if } R > H(X) \,. \end{cases}$$

In this problem, we consider variants of fixed-length compression with side information, i.e., there is a random variable $Y \in \mathcal{Y}$ correlated with the source X that can be used when compressing X. As usual, we write P_{XY} for the joint distribution of X and Y and this distribution is assumed to be known to everybody. Recall that we also write $P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}$.

1. In this question, the compressor and the decompressor have access to the random variable Y. More precisely, a compressor is now $C : \mathcal{X} \times \mathcal{Y} \to \{0,1\}^\ell$ and the decompressor is a function $D : \{0,1\}^\ell \times \mathcal{Y} \to \mathcal{X}$. The error probability is defined as $\mathbf{P}\{D(C(X,Y),Y) \neq X\}$. Note that the probability is over X and Y. Let us call $\delta^{\text{opt}}(X|Y,\ell)$ the smallest error probability over all compressor-decompressor pair.

   (a) Suppose $X = Y$ with probability 1, what can you say on $\delta^{\text{opt}}(X|Y,\ell)$?

   (b) Show that $\delta^{\text{opt}}(X|Y,\ell) = \underset{y\sim P_Y}{\mathbf{E}} \left\{\delta^{\text{opt}}(P_{X|Y=y},\ell)\right\}$

   (c) Using Eq. (1) as a black-box, deduce that we have

   $$\mathbf{P}\left\{\log_2 \frac{1}{P_{X|Y}(X|Y)} > \ell + \tau\right\} - 2^{-\tau} \leq \delta^{\text{opt}}(X|Y,\ell) \leq \mathbf{P}\left\{\log_2 \frac{1}{P_{X|Y}(X|Y)} > \ell\right\} .$$

   (d) If we now take n independent pairs (X_i, Y_i) distributed according to P_{XY}, and let $X^n = X_1 ... X_n$ and $Y^n = Y_1, ..., Y_n$. What can you say on the limit $\lim_{n\to\infty} \delta^{\text{opt}}(X^n|Y^n, Rn)$ for different values of R?

2. Now we consider a setting where the compressor *does not* have access to Y. Only the decompressor sees Y. So the compressor is now $C : \mathcal{X} \to \{0,1\}^\ell$ and $D : \{0,1\}^\ell \times \mathcal{Y} \to \mathcal{X}$. The error probability is given by $\mathbf{P}\{D(C(X),Y) \neq X\}$. We call $\delta^{\text{opt}}_{SW}(X|Y,\ell)$ the smallest error probability for such a compressor-decompressor pair in this setting.

   (a) Using the previous questions, show that

   $$\mathbf{P}\left\{\log_2 \frac{1}{P_{X|Y}(X|Y)} > \ell + \tau\right\} - 2^{-\tau} \leq \delta^{\text{opt}}_{SW}(X|Y,\ell) .$$

   (b) We choose the compressor as follows. For every $x \in \mathcal{X}$, let B_x be uniformly random and independent bitstrings of length ℓ. We set $C(x) = B_x$ for all $x \in \mathcal{X}$. Then define

   $$D(w,y) = \begin{cases} x & \text{if } x \text{ is the unique such that } C(x) = w \text{ and } \log_2 \frac{1}{P_{X|Y}(x|y)} \leq \ell - \tau \\ x_0 & \text{otherwise ,} \end{cases}$$

   for some arbitrary $x_0 \in \mathcal{X}$. Show that in expectation over the choice of B_x for $x \in \mathcal{X}$, the error probability of the pair (C, D) is bounded above by

   $$\mathbf{P}\left\{\log_2 \frac{1}{P_{X|Y}(X|Y)} > \ell - \tau\right\} + 2^{-\tau} .$$

(c) If we now take n independent pairs (X_i, Y_i) distributed according to P_{XY}. What can you say on the limit $\lim_{n\to\infty} \delta_{SW}^{\text{opt}}(X^n|Y^n, Rn)$ for different values of R?

3. (*Advice: Only do this question if you have completed the previous ones*) We now consider a different setting called distributed compression. Suppose Alice compresses X using $C_1 : \mathcal{X} \to \{0,1\}^{\ell_1}$ and Charlie compresses Y using $C_2 : \mathcal{Y} \to \{0,1\}^{\ell_2}$ and the decompressor $D : \{0,1\}^{\ell_1} \times \{0,1\}^{\ell_2} \to \mathcal{X} \times \mathcal{Y}$ received both $C_1(X)$ and $C_2(Y)$ and is asked to recover both X and Y. In this case the error probability of error is given by $\mathbf{P}\{D(C_1(X), C_2(Y)) \neq (X,Y)\}$. We then denote $\delta^{\text{opt}}(X, Y, \ell_1, \ell_2)$ to be the smallest error probability that can be achieved. Take n independent pairs (X_i, Y_i) distributed according to P_{XY}.

(a) Show that if $R_1 > H(X)$ and $R_2 > H(Y|X)$, then the limit

$$\lim_{n\to\infty} \delta^{\text{opt}}(X^n, Y^n, R_1 n, R_2 n) = 0 \ .$$

(b) More generally, what can you say on the set $\mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}_+$ of rates such that $\lim_{n\to\infty} \delta^{\text{opt}}(X^n, Y^n, R_1 n, R_2 n) = 0$ for any $(R_1, R_2) \in \mathcal{R}$? (Do not worry about the boundary $\partial\mathcal{R}$ of $\mathcal{R}$). Try to draw schematically the set $\mathcal{R}$.